A BELIEF-DRIVEN ORDER BOOK MODEL

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1. Introduction

As first pioneered by [19] and treated in detail in [9] one of the crucial roles of market microstructure is price revelation. Different trading systems have emerged and evolved, from auctions to limit order books and dark pools. Each of them correspond to a different way of handling the risk of adverse selection, highlighting the importance played by private beliefs in the trading process. Agents either attempt to protect their own information, or try to infer that of others. This game of cat and mouse between liquidity takers and providers, between informed and uniformed traders highlights the need for models where different beliefs compete on a given market structure. This paper focuses on the limit order book, a microstructure used in more than half the world's exchanges.

1.1. **Classification of models.** Several surveys of microstructure models exist ([17, 9, 5, 10]) but we will follow the methodology of [9] to situate our model. "Market microstructure models can be classified along at least four dimensions: type of orders, sequence of moves, price setting rule, and competitive versus strategic structure." Our model features limit orders and market orders on both the bid and the ask. Traders place both types of orders simultaneously. The pricing is discriminatory: the price of a unit of volume depends on how much has already been traded. Finally, our model is competitive: the interactions between players are of mean-field type, and each agent therefore ignores their own impact when the number of players is large.

The aim of the model is to relate as closely as possible the belief-distribution of the different agents to the shape of the order book. While time-dependences such as price impact, order book resilience and decay of information are also important, we have removed them from the current model, which happens in one period. A direct generalization in continuous time is possible, but the basic intuitions provided will be the same, such that these time-dependence question of the order book will be directly related to time related properties of the agent belief distribution.

This paper was strongly motivated by a series of studies underlining the empirical properties of the limit order book on electronic exchanges ([6, 22, 8, 7, 25, 23, 16, 11]). In particular, the 'hump shape' of the order book was successfully reproduced using reasonable assumptions on the agent belief distribution. The family of models most closely related are informed trader models ([19, 13]): differing beliefs on the price play an important role, although our model does not require noise traders to function. Other approaches focus on inventory risk of the liquidity provider rather

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than on the adverse selection phenomenon ([14, 4]). Another class of models are so called 'zero-intelligence' models ([15, 21, 12, 20, 18]), where strategic considerations are discarded to see which characteristics could be explained by microstructure only. Finally, market impact models([2, 3, 26, 1, 24]) essentially directly model the order book properties without referring to the underlying agents forming it.

1.2. **Results.** The paper is structured in three parts. First the setting of the model is introduced. The agents, the probability space and their differing beliefs are defined. The controls are introduced and their interactions and effects on other variables made explicit. Transaction costs are defined as well as a duality relationship with the order book proved. The section ends by writing up each agent's optimization problem. The second section explores partial equilibrium. The optimal trading strategy is derived first, followed by the limit order posting strategy. These highlight the link between agent beliefs and the distribution of trades and orders. Multiple consequences and several examples conclude the section. The last section searches for Nash equilibria using a mean field game approach. The existence of an equilibrium is derived in the infinite player setting and proved to be an approximate Nash equilibrium in the case of a finite number of agents on the market.

The main contributions are:

- The duality relationship between order book and transaction costs, which defines a codebook for trades.
- The optimality equations and the definition of implied alpha, which provides some economic background to the codebook and links beliefs to trades and orders.
- The Nash equilibrium result and final shape of the order book.

2. Setup

The aim of this section is to introduce the setting and notation to model agents with diverse beliefs engaging in trades on a limit order book. The setup of the model begins with the players and their controls. We introduce the probability space and the order book they interact on. In particular, the different agents' actions lead to transaction costs, which we link back to the original order structure of the market. Next, we describe how these interactions on the order book level affect the players' asset positions and total wealth. The derived equations hold regardless of the number of players, which is reflected in the final notation. Finally, each agent is given an objective function to maximize. The section ends with a summary of each agent's optimization problem.

2.1. **Players and the order book.** We first start describing the players of our game, their beliefs and the setting in which they trade.

Assume there are agents indexed on a set $\mathbb{I}$ with cardinal $n \in [2, \infty]$, each having a control (l^i, g^i) in the control space $\mathbb{R} \times \mathcal{M}(\mathbb{R})$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and p_T a random variable representing the price of an asset. The initial deterministic price is p_0. Each agent possesses a different probability measure $\mathbb{P}^i$ representing his beliefs and which is dominated by the reference measure $\mathbb{P}$. The first component l^i of each agent's control is a real number representing the *signed volume* he wishes to obtain from the market. The second component g^i is a non-negative measure which represents the orders he places on the limit order book[1]. Assume the exchange restricts the orders to be of size one, making each g^i a probability measure. We will also recenter the measures by shifting p_0 to 0. The limit order book is then just the sum of all the orders, which we will renormalize with respect to the total number of agents. If there is a countable number of agents, $g = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g^i$, where the limit is assumed to exist in the weak sense of distributions. All the orders placed on $(0,\infty)$ are offers, and those on $(-\infty,0)$ are bids.

The order book will be split so that each agent will face independent copy g on which they trade simultaneously[2]. When an agent trades a volume l^i on the order book g, he will consume all the orders until he gets filled for the requested volume. This means he will execute all the orders from 0 to the level $f(l^i) = \inf\left\{\alpha \in \mathbb{R} : \int_0^\alpha g = l^i\right\}$, given that an agent will always prefer cheaper orders to more expensive ones. The function f is the generalized inverse of the antiderivative of g. The total amount of money the agent will pay is therefore $p_0 l^i + \int_0^{l^i} f(l)dl$. Because $p_0 l^i$ is the amount he would have paid in the absence of microstructure, $c(l^i) = \int_0^{l^i} f(l)dl$ are the *transaction costs* associated to the trade and represent the extra cost incurred by the agent for taking liquidity from the order book. f is increasing and hence c is a convex function. This chain of computations can be summarized by the following proposition:

Proposition 2.1 (Duality between order book and transaction costs)**.** *Let g be an order book and c the associated transaction costs. Define γ as the Legendre transform of c:*

$$\gamma(\alpha) = \sup_{l \in supp(c)} \alpha l - c(l) \tag{2.1}$$

then we have the relationship

$$\gamma'' = g \tag{2.2}$$

where the second derivative is to be understood in the sense of distributions.

Proof. By construction, c is differentiable and the anti-derivative of g is the generalized inverse of c'. By Fenchel's identity, γ' and c' are generalized inverses of each other. Hence $\gamma' = \int_0^{\cdot} g$, which concludes. □

The following properties are standard results in convex analysis:

Corollary 2.2. *c is a convex, differentiable function with compact support (in the sense of convex functions) and*

$$c(l) + \gamma(\alpha) \geq \alpha l \tag{2.3}$$

for all (α, l) with equality if $c'(l) = \alpha$, or equivalently, $l = \gamma'(\alpha)$.

Proof. g being a non-negative measure with finite mass implies that γ' is non-decreasing and bounded. Its generalized inverse c' is therefore non-decreasing and has compact support. This implies that c is convex and has compact support. The Legendre transform is a bijection for convex functions, making c the Legendre transform of γ. The result follows by Fenchel's inequality. □

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[1] In practice, these are Dirac masses placed on a discrete grid and the resulting order book becomes a histogram of prices.

[2] This is to avoid having to assign priorities. One can also consider that each agent only trades once, and that orders are refilled in between each trade until the new price P_T is revealed.

This allows us to define a codebook for trade volumes. For a given order book, we define $\alpha^i = c'(l^i)$, which also implies the relationship $l^i = \gamma'(\alpha^i)$. Furthermore,

(1) $c(0) = 0$. Agents that do not trade incur no transaction costs.
(2) $c'(0) = 0$. Transaction costs are defined with respect to the current price p_0.
(3) $\gamma(0) = 0$, $\gamma'(0) = 0$. These are dual relationships to (1) and (2).

This defines our players, the order book and by duality their incurred transaction costs.

2.2. **Position and wealth.** This subsection describes more in detail the interactions between the agents' control variables and how those affect their portfolio and wealth.

Each agent simultaneously posts both an order measure $\gamma''^i = g^i$ and a trade volume l^i. The order book is then defined as the renormalized sum of the orders and we likewise define the renormalized measure of trades, $\mu_l = \frac{1}{n}\sum_{i\in\mathbb{I}} \delta_{l^i}$, or equivalently, the measure on the dual variables $\mu = \frac{1}{n}\sum_{i\in\mathbb{I}} \delta_{\alpha^i}$. In the countable agent case, limits must be taken and assumed to exist under the Wasserstein metric. Both are probability measures and we will further assume in the countable case that μ has a finite first moment. For the sake of convenience, we define $\mu^i = \delta_{\alpha^i}$. All orders and trades are considered as simultaneous and the aggregate measures are public knowledge[3].

Let us make more explicit the transfers to agent i triggered by a trade of agent j. Agent j only perceives the aggregate order book γ'' and asks for a volume l^j, or equivalently, executes all order up to the level α^j. The volume l^j verifies the duality relationship $l^j = \gamma'(\alpha^j) = \langle \gamma', \mu^j \rangle$. The transfer from player j to player i however, will be equal to $l^{ji} = \frac{1}{n}\gamma'^i(\alpha^j) = \langle \gamma'^i, \mu^j \rangle$[4]. This corresponds to all the orders of player i that are placed between 0 and α^j. The additivity of the order book ensures that $l^j = \frac{1}{n}\sum_{i\in\mathbb{I}} l^{ji}$ (with a limit in the countable case).

Similarly, agent j's cash transfer for trading l^j is $c\left(l^j\right) = \alpha^j l^j - \gamma(\alpha^j)$ by the duality relationship. Expressing everything in terms of the dual variables leads to the formula $c(l^j) = \langle id\gamma' - \gamma, \mu^j \rangle$. The following lemma introduces a convenient adjoint operator.

Lemma 2.3. *For any limit order book γ'' and any probability measure μ of order 1 with a finite first moment, we have*

$$\langle (id - \alpha^*)\gamma' - \gamma, \mu \rangle = \langle \gamma'', \mathcal{L}_{\alpha^*}\mu \rangle \tag{2.4}$$

where we define

$$\mathcal{L}_{\alpha^*}\mu = (id - \alpha^*)\left(\int_{\cdot}^{\infty} \mu - 1_{\cdot\le 0}\right) \tag{2.5}$$

Proof. $\mathcal{L}_{\alpha^*}\mu$ is a well defined function because $\left|(\alpha - \alpha^*)\int_\alpha^\infty \mu\right| \le |\alpha^*| + \int_{\mathbb{R}} |\text{id}|\mu$ for $\alpha > 0$. A similar result holds when $\alpha \le 0$. Successive integrations by parts and

[3]The exact rules of what is and is not known depends on the specific market considered. On NASDAQ, orders can be traced to individual agents, but not trades. In Foreign Exchange, sometimes both orders and trades can be traced back to market participants.

[4]Remember that each agent's order is split equally into one of the n independent copies of the aggregate order book.

simplifications then yield:

$$\begin{aligned}\left\langle \gamma'', (id-\alpha^*)\left(\int_{\cdot}^{\infty} \mu - 1_{\cdot\leq 0}\right)\right\rangle &= -\left\langle \gamma', \int_{\cdot}^{\infty} \mu - 1_{\cdot\leq 0} - (id-\alpha^*)\mu - \alpha^*\delta_0\right\rangle \\ &= \langle \gamma, \mu - \delta_0\rangle + \langle \gamma'(id-\alpha^*), \mu\rangle + \alpha^*\gamma'(0) \\ &= \langle (id-\alpha^*)\gamma' - \gamma, \mu\rangle\end{aligned}$$

where the first integration by parts is justified by the fact that γ' is bounded, the Dirac distribution has compact support and the equations

$$\lim_{\alpha\to\infty}\int_{\alpha}^{\infty} \mu = 0 \tag{2.6}$$

$$\lim_{\alpha\to-\infty}\int_{\alpha}^{\infty} \mu - 1_{\alpha\leq 0} = -\lim_{\alpha\to-\infty}\int_{-\infty}^{\alpha} \mu = 0 \tag{2.7}$$

$$\tag{2.8}$$

and $\alpha\mu(\alpha)$ vanishes at the infinities. The linear growth of γ justifies the second integration by parts. The last line uses the initial conditions of γ and γ'. □

This allows us to rewrite the transaction costs as $c(l^j) = \left\langle \gamma'', \mathcal{L}_0\mu^j\right\rangle$. The transfer of money from agent j to i is $\frac{1}{n}\left\langle \gamma''^i, \mathcal{L}_0\mu^j\right\rangle$.

The position of agent i is therefore

$$L^i = \left\langle \gamma', \mu^i\right\rangle - \left\langle \gamma'^i, \mu\right\rangle \tag{2.9}$$

While the wealth of agent i at time T is

$$X^i = L^i\left(p_T - p_0\right) + \left\langle \gamma''^i, \mathcal{L}_0\mu\right\rangle - \left\langle \gamma'', \mathcal{L}_0\mu^i\right\rangle \tag{2.10}$$

where these formulas are hence well-defined both in the finite and infinite agent case.

What is important to note is that all the interaction between the agents are of *mean field* type. Indeed, each agent interacts with all the others only through his own control variables and the *aggregate* variables μ and γ. This is what allows the notation to extend naturally from the finite to the infinite case. It is also what will allow us in the last section to use the infinite player limit as an approximation to the finite player game.

2.3. **Objective function.** The setup of the game ends with the definition of each agent's objective function. We then summarize the game from the beginning: the players, the control and state variables and the objective function.

Each agent now has a utility function $U^i\left(X^i, p_T\right)$ which depends on his wealth and the price[5]. Assume that each γ''^i is constrained to be a probability measure. Assume U^i to be differentiable and concave in its first variable and the controls chosen such as to verify the integrability conditions

$$\mathbb{E}_{\mathbb{P}^i} U^i\left(X^i, p_T\right) < \infty \tag{2.11}$$

$$0 < \mathbb{E}_{\mathbb{P}^i}\partial_X U^i\left(X^i, p_T\right) < \infty \tag{2.12}$$

Finally, define

$$J^i\left(\alpha^i, \gamma''^i, \mu, \gamma\right) = \mathbb{E}_{\mathbb{P}^i} U^i\left(X^i, p_T\right) \tag{2.13}$$

as agent i's *objective function.*

[5]This covers Merton-type problems, indifference pricing and statistical arbitrage.

To summarize, each agent therefore faces the optimization problem:

$$\max_{(\alpha^i,\gamma''^i)} \mathbb{E}_{\mathbb{P}^i} U^i\left(X^i, p_T\right) \tag{2.14}$$

subject to

$$\begin{cases} L^i &= \left\langle \gamma', \mu^i \right\rangle - \left\langle \gamma'^i, \mu \right\rangle \\ X^i &= L^i\left(p_T - p_0\right) + \left\langle \gamma''^i, \mathcal{L}_0\mu \right\rangle - \left\langle \gamma'', \mathcal{L}_0\mu^i \right\rangle \end{cases} \tag{2.15}$$

where $\mu^i = \delta_{\alpha^i}$ and γ''^i must be a probability measure.

3. Partial equilibrium

In this section, we will focus on one particular player i. To maintain the mean-field structure of our game, we assume the price to be a (random) function only of the aggregate quantities μ and γ. In particular, any price impact is symmetric amongst players, and in the infinite player limit, each agent's price impact is negligeable. In partial equilibrium, we will assume our agent has been given the price exogenously, which will be strictly true in the infinite player limit and serve as a stepping stone in the finite player equilibrium. We first derive the optimal strategy in terms of trades and order placement for our agent. Then we study some consequences of collective optimal behavior.

3.1. **Optimality equations.** The agent's optimization problem is straightforward to solve. The optimal trading strategy is obtained by differentiating the objective function in l^i. The order placing strategy is derived from the affine nature of the objective function in γ''^i, forcing the optimal solution to be an extremal point of our permissible set.

The first result gives us a natural way of naming the dual variable α^i:

Theorem 3.1 (Implied alpha)**.** *For a given μ and γ, the optimal trading strategy of player i is*

$$\alpha^i = \mathbb{E}_{\mathbb{Q}^i}\left[p_T - p_0\right] \tag{3.1}$$

where $\frac{d\mathbb{Q}^i}{d\mathbb{P}^i} = \frac{\partial_X U^i\left(X^i, p_T\right)}{\mathbb{E}_{\mathbb{P}^i}\partial_X U^i(X^i, p_T)}$.

Proof. This result is more easily shown with the primal variable l^i. Rewriting the state variables yields $L^i = l^i - \left\langle \gamma'^i, \mu \right\rangle$ and $X^i = L^i(p_T - p_0) + \left\langle \gamma''^i, \mathcal{L}_0\mu \right\rangle - c(l^i)$. Note that X^i is a concave function of l^i.

Because U^i is concave and increasing in X, $\mathbb{E}_{\mathbb{P}^i}\left[U^i\left(X^i, p_T\right)\right]$ is a concave function of l^i. We can therefore simply equate the derivative to zero. This yields the equation:

$$\mathbb{E}_{\mathbb{P}^i}\left[\left(p_T - p_0 - c'(l^i)\right)\partial_X U^i\left(X^i, p_T\right)\right] = 0 \tag{3.2}$$

which simplifies to

$$\mathbb{E}_{\mathbb{P}^i}\left[\partial_X U^i\left(X^i, p_T\right)\right] c'(l^i) = \mathbb{E}_{\mathbb{P}^i}\left[\left(p_T - p_0\right)\partial_X U^i\left(X^i, p_T\right)\right] \tag{3.3}$$

The expectation on the left hand side is positive and finite by assumption, allowing us to do the announced change of measure. The identity $c'(l^i) = \alpha^i$ then concludes. □

The quantity $p_T - p_0$ corresponds to what practitioners call the *realized alpha* of the trade. Motivated by the above result, we name the dual variable α^i the *implied alpha* of the trade. The theorem states that the implied alpha is equal to the agent's expectation of the *realized alpha* under some new probability measure

$\mathbb{Q}^i$. Because we have that $\mathbb{Q}^i = \mathbb{P}^i$ in the risk-neutral case, we will name $\mathbb{Q}^i$ the *risk-neutral* probability measure of agent i.

The second result links the agent's orders to μ, the aggregate distribution of implied alphas.

Proposition 3.2 (Profitability of an order)**.** *For a given μ and γ, player i's optimal strategy for orders is*

$$\gamma''^i = \delta_{\tilde{\alpha}^i} \tag{3.4}$$

where

$$\tilde{\alpha}^i = \textit{argmax}\ \mathcal{L}_{\alpha^i}\mu \tag{3.5}$$

Proof. First we verify that the argmax is well defined. The function $m = \mathcal{L}_{\alpha^i}\mu$ is continuous on $(-\infty, 0]$ and $(0, +\infty)$ and vanishes at the infinities. Furthermore, both $m(0^+)$ and $m(0)$ are finite and of opposing signs. Hence one of them is positive and m reaches its supremum either at a point of continuity, 0 or 0^+.[6]

Let all the variables but γ''^i be fixed. Denote by $X^i\left(\gamma''^i\right)$ the affine functional

$$\gamma''^i \longrightarrow \left\langle \gamma''^i, \mathcal{L}_{p_T - p_0}\mu \right\rangle - \left\langle \gamma'', \mathcal{L}_{p_T - p_0}\mu^i \right\rangle \tag{3.6}$$

The concavity of U^i implies that

$$\Delta J := \mathbb{E}_{\mathbb{P}^i}\left[U^i\left(X^i(\gamma''^i), p_T\right) - U^i\left(X^i(\delta_{\tilde{\alpha}^i}), p_T\right)\right] \tag{3.7}$$

verifies

$$\begin{aligned}
\Delta J &\leq \mathbb{E}_{\mathbb{P}^i}\left[\left\langle \gamma''^i - \delta_{\tilde{\alpha}^i}, \mathcal{L}_{p_T - p_t}\mu \right\rangle \partial_X U^i\left(X^i(\gamma''^i), p_T\right)\right] \\
&\leq \mathbb{E}_{\mathbb{P}^i}\left[\partial_X U^i\left(X^i(\gamma''^i), p_T\right)\right] \mathbb{E}_{\mathbb{Q}^i}\left\langle \gamma''^i - \delta_{\tilde{\alpha}^i}, \mathcal{L}_{p_T - p_0}\mu \right\rangle \\
&= \mathbb{E}_{\mathbb{P}^i}\left[\partial_X U^i\left(X^i(\gamma''^i), p_T\right)\right] \left\langle \gamma''^i - \delta_{\tilde{\alpha}^i}, \mathcal{L}_{\alpha^i}\mu \right\rangle
\end{aligned}$$

The first term is positive by hypothesis. The second term is non-positive as

$$\begin{aligned}
\left\langle \gamma''^i, \mathcal{L}_{\alpha^i}\mu \right\rangle &\leq ||\mathcal{L}_{\alpha^i}\mu||_\infty \\
&= \langle \delta_{\tilde{\alpha}^i}, \mathcal{L}_{\alpha^i}\mu \rangle
\end{aligned}$$

where we have used the fact that γ''^i is a probability measure. This concludes. □

The operator $\mathcal{L}_{\alpha^i}$ quantifies the *expected profitability* of an order. Indeed,

$$\mathcal{L}_{\alpha^i}\mu(\alpha) = \underbrace{\left(\alpha - \alpha^i\right)}_{\textit{exp. spread}} \underbrace{\left(\int_\alpha^\infty \mu - 1_{\alpha \leq 0}\right)}_{\textit{filling probability}} \tag{3.8}$$

is the expected gain under $\mathbb{Q}^i$ of an order placed at the level α. The absolute value of the first term is equal to the *spread* the liquidity provider expects to gain per filled order. The second term, up to a sign, is equal to the proportion of agents that will fill the order. If the agents of our model arrived according to Poisson arrival times instead of simultaneously, this would be the *filling probability* of the order. These two terms are commonly used by practitioners.

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[6]The difference between 0^+ and 0^- is that one is a buy and the other one a sell order at the current price p_0.

3.2. **Corollaries.** The objective of this subsection is to explore some consequences of the above results when adding additional hypotheses. We start with a result linking the average implied alpha to the market risk premium.

Corollary 3.3 (Market implied price)**.** *Assume $p_T - p_0$ to be bounded a.s. and all the agents to be exchangeable*[7] *under $\mathbb{P}$. Then we have*

$$\langle id, \mu \rangle = \mathbb{E}_{\mathbb{P}} \left[p_T - p_0 \right] \tag{3.9}$$

and the inequality

$$\langle f, \mu \rangle \leq \mathbb{E}_{\mathbb{P}} f(p_T - p_0) \tag{3.10}$$

for f a convex function such that $f(p_T - p_0)$ is a.s. bounded.

Proof. We have the chain of relationships

$$\begin{aligned} \langle f, \mu \rangle &= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\mathbb{E}_{\mathbb{P}^i} \left[p_T - p_0 \right] \right) & (3.11) \\ &\leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\mathbb{P}} \left[Z^i f\left(p_T - p_0 \right) \right] & (3.12) \\ &= \mathbb{E}_{\mathbb{P}} \left[f\left(p_T - p_0 \right) \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Z^i \right] & (3.13) \\ &= \mathbb{E}_{\mathbb{P}} f(p_T - p_0) & (3.14) \end{aligned}$$

where $Z^i = \frac{d\mathbb{Q}^i}{d\mathbb{P}}$. Fubini's theorem in (3.13) is justified because $f\left(p_T - p_0 \right)$ is bounded, and the Z^i are uniformly bounded. (3.12) is an application of Jensen's inequality and is an equality for $f = id$. The exchangeability assumption allows us to apply the law of large numbers to the Z^i which results in (3.14). □

The above result suggests a very intuitive model of the dependence of p_T in μ:

$$p_T = p_0 + \langle id, \mu \rangle + N \tag{3.15}$$

with N of mean zero. The empirical average of the implied alpha is therefore equal to the *market risk premium*. If N is now exogeneous, then p_T is uniformly Lipschitz under the Wasserstein metric, which will matter to find an equilibrium.

The corollary also implies that μ is not quite a pricing measure, as it underestimates events far away from the mean. Amusingly, γ'' does exactly the opposite, pushing away weight from the average scenario into the periphery.

Corollary 3.4 (The double hump shape of the order book)**.** *Assume μ has a positive, differentiable L^2 density and that agents are constrained to place orders on the same side as their trades. Assume furthermore $-\mu' \int_\alpha^\infty \mu < 2\mu^2$ for $\alpha > 0$ (and the symmetric relationship for $\alpha < 0$). Then α^i and $\tilde{\alpha}^i$ verify*

$$\mu(\tilde{\alpha}^i) \left(\tilde{\alpha}^i - \alpha^i \right) = \int_{\tilde{\alpha}^i}^{\pm\infty} \mu \tag{3.16}$$

[7] That is, consider all their intrinsic variables, utility functions and Radon-Nikodym derivatives $\frac{d\mathbb{P}^i}{d\mathbb{P}}$ to be assigned in an exchangeable manner at the beginning of the game.

where $\int_{\tilde{\alpha}^i}^{\pm\infty} \mu = \int_{\tilde{\alpha}^i}^{\infty} \mu$ *if* $\alpha^i > 0$ *and* $\int_{\tilde{\alpha}^i}^{\pm\infty} \mu = -\int_{-\infty}^{\tilde{\alpha}^i} \mu$ *else. In particular, agent* i*'s spread is*

$$\frac{1}{\mu(\tilde{\alpha}^i)} \left| \int_{\tilde{\alpha}^i}^{\pm\infty} \mu \right| \tag{3.17}$$

Proof. Without loss of generality we can assume $\alpha^i > 0$. The additional constraint implies that

$$\tilde{\alpha}^i = argmax_{\alpha>0} m(\alpha) \tag{3.18}$$

where $m = \mathcal{L}_{\alpha^i}\mu$. We have that $m(0) < 0$, $m(\alpha^i) = 0$ and $m(\infty) = 0$. Furthermore, m is differentiable and

$$m'(\alpha) = \int_{\alpha}^{\infty} \mu - \left(\alpha - \alpha^i\right) \mu(\alpha) \tag{3.19}$$

$m'(0^+) = \int_0^{\infty} \mu + \alpha^i \mu(0) > 0$ and $m(\infty) = 0$ as μ has finite first moment. Because $\mu > 0$, the function

$$M(\alpha) = \frac{1}{\mu(\alpha)} \int_{\alpha}^{\infty} \mu - \alpha + \alpha^i \tag{3.20}$$

has the same sign as $m'(\alpha)$. Differentiating yields:

$$M'(\alpha) = -\frac{\mu(\alpha)^2 + \mu'(\alpha) \int_{\alpha}^{\infty} \mu}{\mu(\alpha)^2} - 1 \tag{3.21}$$

which is negative by hypothesis. Hence M is decreasing and m' has a unique zero. $m'(0^+) > 0$ and hence m has a unique maximum $\tilde{\alpha}^i$, which verifies $m'(\tilde{\alpha}^i) = 0$. This concludes. □

This provides us with a methodology to derive γ'' from μ. The inverse problem is of course harder, but still possible for parametric choices of μ. It is easy to verify numerically that, for a unimodal 'bell-shaped' μ probability measure, γ'' is a bimodal 'double-hump' distribution. This coincides with empirical studies of the limit order book.

Example 3.5. *In the case of a Laplace distribution of parameter* λ

$$\mu(\alpha) = \frac{\lambda}{2} e^{-\lambda|\alpha|} \tag{3.22}$$

The Laplace distribution verifies verifies all the hypotheses except differentiability at 0*. The spread is then constant equal to*

$$\frac{1}{\mu(\tilde{\alpha}^i)} \left| \int_{\tilde{\alpha}^i}^{\pm\infty} \mu \right| = \frac{1}{\lambda} \tag{3.23}$$

and γ'' *is a Laplace distribution that is* shifted $\frac{1}{\lambda}$ *away from* 0 *on both sides. In this case there is a well defined best bid and best ask, and the bid-ask spread is equal to* $\frac{2}{\lambda}$*. This distribution exhibits a peak of liquidity at the best bid and ask, contrary to what is commonly seen on the market.*

Example 3.6. *The standard Gaussian distribution verifies all the hypotheses. The spread does not have an explicit formula, but can be computed numerically. In this case, the best bid and ask are limited only by the granularity of the market and the order book has a peak of liquidity behind the best bid and best ask.*

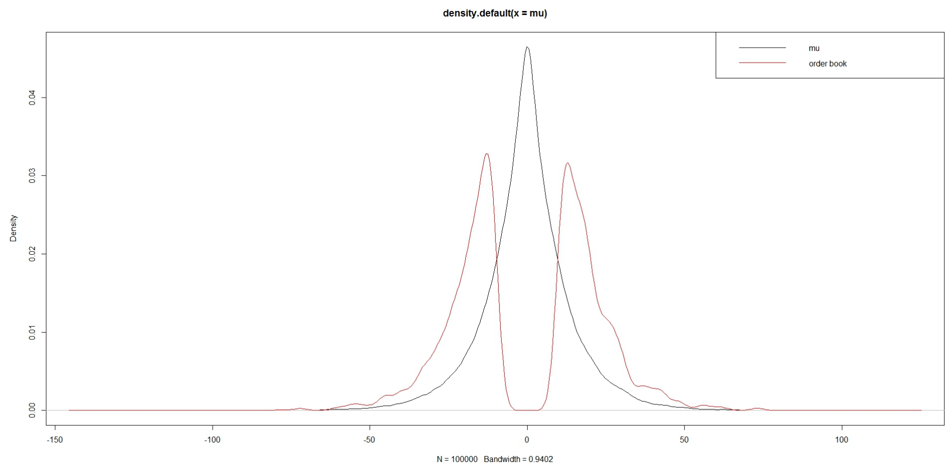

FIGURE 1. Simulated double-hump shape of the order book in the case of a Laplace distribution for μ.

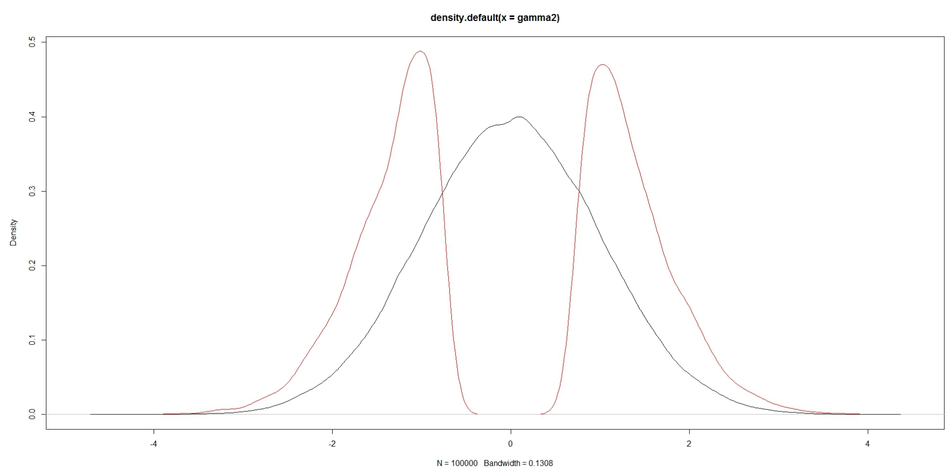

FIGURE 2. Simulated double-hump shape of the order book in the case of a Gaussian distribution for μ.

The above optimality equations and their consequences illustrate the usefulness of the proposed codebook for trades and naturally fits with terms commonly used in the industry. They can be derived without too much difficulty in a continuous time framework, even in a setting where agents have differing time horizons and filtrations.

4. NASH EQUILIBRIA

The natural notion of equilibrium to pursue in our game is of Nash type. The mean field nature of the interactions between players allowed us to derive simple optimality equations in partial equilibrium. Unfortunately, these only hold as long as each agent ignores his own price impact. This will be true in the limit where the number of players is infinite, but will only approximately hold for a finite but large number of players. The first subsection therefore explores the infinite player case. The fixed point theorem proved requires some strong assumptions: in particular, the agents are assumed to be risk-neutral. In the second subsection the infinite player strategy is given to players in a finite setting and proved to be an *approximate* Nash equilibrium. In both cases, only questions of existence are explored.

4.1. **The infinite player limit.** By taking the limit as the number of players goes to infinity, the problem gains in tractability by making everything happen 'in a continuum'. It also guarantees the absence of price impact for any single agent, although the players possess, in aggregate, impact over the market.

Without loss of generality, we can assume that $p_0 = 0$ and replace each agent's second variable γ''^i by $\tilde{\alpha}^i$ with the relationship $\gamma''^i = \delta_{\tilde{\alpha}^i}$. Furthermore assume that p_T lives in the compact set $[-K, K]$ and that

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} = Z(\theta^i) \tag{4.1}$$

with Z uniformly continuous in θ and the θ^i such that, as $n \to \infty$ the empirical distributions $\nu_n = \frac{1}{n}\sum_{j=1}^n \delta_{\theta^j}$ and $\nu_n^i = \frac{1}{n}\sum_{j\neq i}^n \delta_{\theta^j}$ converge to a probability measure ν. Assume ν has a continuous density that we abusively also denote by ν. Define the functional

$$\begin{aligned} J(\alpha, \tilde{\alpha}, \mu, \gamma'', \theta) &= \mathbb{E}\left[Z(\theta)\left(\mathcal{L}^*_{p_T(\mu)}\gamma''(\alpha) - \mathcal{L}_{p_T(\mu)}\mu(\tilde{\alpha})\right)\right] \\ &= \mathcal{L}^*_{p(\theta,\mu)}\gamma''(\alpha) - \mathcal{L}_{p(\theta,\mu)}\mu(\tilde{\alpha}) \end{aligned}$$

with $p(\theta, \mu) = \mathbb{E}\left[Z(\theta)p(\mu)\right]$. Assume p is continuous in the second argument with respect to the Wasserstein metric, uniformly over the first argument. Then the objective function for player i in the finite agent case is equal to:

$$\begin{aligned} J^i\left(\left(\alpha^j\right)_{j=1..n}, (\tilde{\alpha})_{j=1..n}\right) &= J(\alpha^i, \tilde{\alpha}^i, \mu_n, \gamma''_n, \theta^i) \\ &= J(\alpha^i, \tilde{\alpha}^i, \mu_n^i + \frac{1}{n}\delta_{\alpha^i}, \gamma''^i_n + \frac{1}{n}\delta_{\tilde{\alpha}^i}, \theta^i) \end{aligned}$$

where $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{\alpha^j}$ and $\mu_n^i = \frac{1}{n}\sum_{j\neq i}^n \delta_{\alpha^j}$. γ''_n and γ''^i_n are defined similarly. The idea of the mean field game methodology is to take the limit as $n \to \infty$, making the dependence of J in the player's control disappear from the third and fourth variable. The following theorem shows the existence of a solution to the limiting problem.

Theorem 4.1 (Infinite player problem)**.** *Define the Hilbert space $L^2(\nu)$ as the functions $f : \mathbb{R} \to \mathbb{R}$ that verify $\int f^2\nu < \infty$. Then there exists a pair of functions α_ν and $\tilde{\alpha}_\nu$ in $L^2(\nu)$ such that*

$$\left(\alpha_\nu, \tilde{\alpha}_\nu\right)(\theta) = \mathit{argmax}_{a,\tilde{a}} J(a, \tilde{a}, \mu, \gamma'', \theta)\ \ \nu\text{-a.s.} \tag{4.2}$$

and

$$\mu = \nu \circ \alpha_\nu^{-1}\ \ ;\ \ \gamma'' = \nu \circ \tilde{\alpha}_\nu^{-1} \tag{4.3}$$

Proof. By (3.1), we need to find a fixed point to the map

$$M\alpha(\theta) = p(\theta, \nu \circ \alpha^{-1}) \tag{4.4}$$

as the optimality condition for $\tilde{\alpha}$ just reads

$$\tilde{\alpha}(\theta) = argmax\ \mathcal{L}_{\alpha(\theta)}\mu \tag{4.5}$$

which does not require a fixed point once α is solved for. By assumption, $||M(\alpha)||_\infty \leq K$. The ball of radius K is therefore stable and weakly compact with respect to the considered Hilbert norm. Hence, all we need to show is the weak continuity of M to conclude by Schauder's fixed point theorem. First, note that if α_n converges weakly to α, then

$$\int |f(\alpha_n(\theta)) - f(\alpha(\theta))|\, \nu(\theta)d\theta \leq \int |\alpha_n(\theta) - \alpha(\theta)|\, \nu(\theta)d\theta \to 0 \tag{4.6}$$

for any 1-Lipschitz function f. Hence $\nu \circ \alpha_n^{-1} \to \nu \circ \alpha^{-1}$ under the Wasserstein metric. This concludes. □

Corollary 4.2 (Nash equilibrium)**.** *The strategy* $\big(\alpha(\theta^i), \tilde{\alpha}(\theta^i)\big)_{i\in\mathbb{I}}$ *is a Nash equilibrium.*

Proof. By construction, $\big(\alpha(\theta^i), \delta_{\tilde{\alpha}^i}\big)$ verifies the optimality equations for player i when given the aggregate distributions $\mu = \nu \circ \alpha^{-1}$ and $\gamma'' = \nu \circ \tilde{\alpha}^{-1}$. Hence we have that for all $(\beta, \tilde{\beta})$

$$J(\beta, \tilde{\beta}, \nu \circ \alpha^{-1}, \nu \circ \tilde{\alpha}^{-1}, \theta^i) \leq J(\alpha(\theta^i), \tilde{\alpha}(\theta^i), \nu \circ \alpha^{-1}, \nu \circ \tilde{\alpha}^{-1}, \theta^i) \tag{4.7}$$

□

Because we used Schauder's fixed point theorem, uniqueness of the equilibrium remains elusive. Furthermore, the absence of continuity of the second optimality equation makes it hard to treat the case of non-risk neutral agents. On a positive note, however, the methodology can be extended to the continuous time case using the mean field game paradigm, as long as the price impact function remains of mean field type.

4.2. **Approximate Nash equilibrium.** Next, we want to use the previous solution as a strategy in the finite player case. The following lemma will help us show that it leads to an approximate Nash equilibrium.

Lemma 4.3. *For given* α^i*,* $\tilde{\alpha}^i$*,* θ^i*,* ν_n *and* ν *let*

$$e_n = J(\alpha^i, \tilde{\alpha}^i, \nu_n^i \circ \alpha^{-1} + \frac{1}{n}\delta_{\alpha^i}, \nu_n^i \circ \tilde{\alpha}^{-1} + \frac{1}{n}\delta_{\tilde{\alpha}^i}, \theta^i) - J(\alpha^i, \tilde{\alpha}^i, \nu \circ \alpha^{-1}, \nu \circ \tilde{\alpha}^{-1}, \theta^i) \tag{4.8}$$

Then there exists $\epsilon_n \geq 2|e_n|$ *such that* $\epsilon_n \to 0$ *as* $n \to \infty$*,uniformly over* α^i*,* $\tilde{\alpha}^i$ *and* θ^i*.*

Proof. By definition, the first term is equal to

$$\left\langle \mathcal{L}_{p(\theta^i, \nu_n^i \circ \alpha^{-1} + \frac{1}{n}\delta_{\alpha^i})}\delta_{\alpha^i}, \nu_n^i \circ \tilde{\alpha}^{-1} + \frac{1}{n}\delta_{\tilde{\alpha}^i} \right\rangle - \left\langle \mathcal{L}^*_{p(\theta^i, \nu_n^i \circ \alpha^{-1} + \frac{1}{n}\delta_{\alpha^i})}\delta_{\tilde{\alpha}^i}, \nu_n^i \circ \alpha^{-1} + \frac{1}{n}\delta_{\alpha^i} \right\rangle$$
$$= \left\langle \mathcal{L}_{p(\theta^i, \nu_n^i \circ \alpha^{-1} + \frac{1}{n}\delta_{\alpha^i})}\delta_{\alpha^i}, \nu_n^i \circ \tilde{\alpha}^{-1} \right\rangle - \left\langle \mathcal{L}^*_{p(\theta^i, \nu_n^i \circ \alpha^{-1} + \frac{1}{n}\delta_{\alpha^i})}\delta_{\tilde{\alpha}^i}, \nu_n^i \circ \alpha^{-1} \right\rangle$$

By the uniform continuity of p in its second variable, $\mathcal{L}_{p(\theta^i, \nu_n^i \circ \alpha^{-1} + \frac{1}{n}\delta_{\alpha^i})}\delta_{\alpha^i}$ converges weakly to the function $\mathcal{L}_{p(\theta^i, \nu \circ \alpha^{-1})}\delta_{\alpha^i}$. That the convergence is uniform over θ^i is obvious and the uniformity over α^i can be derived from the explicit formula of $\mathcal{L}_p\delta_{\alpha^i}(a)$. By the portmanteau theorem, as ν is continuous, $\langle \alpha \circ \mathcal{L}_{p(\theta^i, \nu \circ \alpha^{-1})}\delta_{\alpha^i}, \nu_n^i \rangle$ converges to $\langle \alpha \circ \mathcal{L}_{p(\theta^i, \nu \circ \alpha^{-1})}\delta_{\alpha^i}, \nu \rangle$. This concludes. □

Finally,

Theorem 4.4 (Approximate Nash equilibrium). *The strategy* $\left(\alpha(\theta^i), \tilde{\alpha}(\theta^i)\right)_{i=1..n}$ *is an* ϵ_n*-Nash equilibrium. That is, for all* $(\beta, \tilde{\beta})$ *and all* $i \in \{1...n\}$ *we have that*

$$J(\beta, \tilde{\beta}, \nu_n^i \circ \alpha^{-1} + \frac{1}{n}\delta_\beta, \nu_n^i \circ \tilde{\alpha}^{-1} + \frac{1}{n}\delta_{\tilde{\beta}}, \theta^i) \leq J(\alpha(\theta^i), \tilde{\alpha}(\theta^i), \nu_n \circ \alpha^{-1}, \nu_n \circ \tilde{\alpha}^{-1}, \theta^i) + \epsilon_n \tag{4.9}$$

Proof. By the previous lemma, we have the chain of inequalities

$$\begin{aligned} J(\beta, \tilde{\beta}, \nu_n^i \circ \alpha^{-1} + \frac{1}{n}\delta_\beta, \nu_n^i \circ \tilde{\alpha}^{-1} + \frac{1}{n}\delta_{\tilde{\beta}}, \theta^i) &\leq J(\beta, \tilde{\beta}, \nu \circ \alpha^{-1}, \nu \circ \tilde{\alpha}^{-1}, \theta^i) + \frac{1}{2}\epsilon_n \\ &\leq J(\alpha(\theta^i), \tilde{\alpha}(\theta^i), \nu \circ \alpha^{-1}, \nu \circ \tilde{\alpha}^{-1}, \theta^i) + \frac{1}{2}\epsilon_n \\ &\leq J(\alpha(\theta^i), \tilde{\alpha}(\theta^i), \nu_n \circ \alpha^{-1}, \nu_n \circ \tilde{\alpha}^{-1}, \theta^i) + \epsilon_n \end{aligned}$$

□

5. Conclusion

This paper presents a one time-step model designed to link the shape of the order book to the heterogenous beliefs of the large number of agents trading on it. In particular, it proposes a codebook for trades that, in equilibrium, summarizes the agents' beliefs on the price. The new variable introduced, dubbed 'implied alpha' to echo the regular notion of alpha, is readily obtained from transaction costs using a duality relationship. The framework is then used to shed some light on coherent price impact, price formation and order book shaping mechanisms. Using ideas borrowed from mean field games, the question of existence of Nash equilibria is answered positively in the infinite player limit. Existence of approximate Nash equilibria in the finite player case is also guaranteed.

A natural extension of the framework would be to answer similar questions in continuous time. Some work has been made to solve the partial equilibrium in a Gaussian filtration setting with heterogeneous filtrations and probability measures on a given price process. Although all these additional details do not fundamentally change the proof method, they do enrich the model significantly by introducing more trade opportunities into the market. Differing time horizons for the agents can also add some realism -and additional trading opportunities- without too much hassle. The question of Nash equilibrium requires more thought as the mean field nature of the game becomes a dominant feature of the continuous time model.

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