Risk-Neutral Models for Emission Allowance Prices and Option Valuation

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Abstract

The existence of mandatory emission trading schemes in Europe and the US, and the increased liquidity of trading on futures contracts on CO_2 emissions allowances, led naturally to the next step in the development of these markets: these futures contracts are now used as underliers for a vibrant derivative market. In this paper, we give a rigorous analysis of a simple risk-neutral reduced-form model for allowance futures prices, demonstrate its calibration to historical data, and show how to price European call options written on these contracts.

Key words Emission derivatives, Emissions markets, Cap-and-trade schemes, Environmental Finance.

1 Introduction

Global warming and environmental problems continue to challenge policy makers. In part because of the success of the U.S. Acid Rain Program, cap-and-trade systems are now considered as one of the most promising market mechanism to reduce Green House Gas (GHG) emissions on an international scale. The core principle of such a mechanism is based on the allocation of fully tradable credits among emission sources and sets a penalty to be paid per unit of pollutant which is not offset by a credit at the end of a pre-determined period. The idea is that the introduction of emission trading leads to price discovery which helps identify and to exercise the cheapest emission abatement measures. For this reason, market-based mechanisms for emission reduction are supposed to yield pollution control at the lowest cost for the society. Notwithstanding the fact that the rigorous equilibrium analysis from [6] and [5] confirm that social optimality does not necessarily mean that the scheme is cheap for consumers, emission trading should be considered as a cost-efficient and effective tool.

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By its very nature, the regulatory framework of any mandatory cap-and-trade system involves its participants in a risky business, necessarily creating the need for appropriate risk management. Trading of certificates from a mandatory scheme is typically accompanied by an active secondary market where diverse emission-related financial assets (futures) including a fast-growing variety of their derivatives are traded. Options have been traded since 2006 and as we explain below on any given day, the volume of European call and put options traded on forward EUA contracts ranges between 15m and 25m tons of CO_2 equivalent, the short end of the curve being clearly the most actively traded.

In this work, we propose reduced form models for the risk-neutral dynamics of allowance prices, providing a quantitative framework for pricing emission derivatives.

Despite the large number of pieces in the popular press and numerous speculative articles in magazines, the scientific literature on cap-and-trade systems is rather limited, especially if one restricts ourselves to quantitative analysis. For the sake of completeness, we briefly review the publications related to our contribution. The *economic theory* of allowance trading can be traced back to [10] and [16] whose authors proposed a market model for the public good *environment* described by tradable permits. *Dynamic allowance trading* is addressed in [9], [24], [18], [14], [19], [22], [15] and in the literature cited therein. *Empirical evidence from existing markets* is discussed in [11] and [17]. This paper suggests economic implications and hints at several ways to model spot and futures allowance prices, whose detailed inter-relations are investigated in [25] and [26]. There, the demand for derivative instruments in emission markets is also addressed. In [2] characteristic properties for financial time series are observed for prices of emission allowances from the mandatory European Scheme EU ETS. Furthermore, a Markov switch and AR-GARCH models is suggested. The work [17] considers also tail behavior and the heteroscedastic dynamics in the returns of emissions allowance prices. *Dynamic price equilibrium and optimal market design* are investigated in [6] which provides a mathematical analysis of the market equilibrium and uses optimal stochastic control to show social optimality. Based on this approach, [5] discusses price formation for goods whose production is affected by emission regulations. In this setting, an equilibrium analysis confirms the existence of the so-called windfall profits (see [21]) and provides quantitative tools to analyze alternative market designs, which are applied in [4] to optimize a cap-and-trade mechanism for a proposed Japanese emission trading scheme. [20] and the PhD thesis [27] also deal with risk-neutral allowance price formation within EU ETS. Using equilibrium properties, the price evolution is treated in terms of marginal abatement costs and optimal stochastic control. *Option pricing* within EU ETS was considered only recently. [7] uses hidden Markov models and a filtering approach to capture the impact of news releases while [8] relies on endogenous emission permit price dynamics within an equilibrium setting to value European option on emission allowances.

The present paper is organized as follows. After an introductory discussion of the various approaches to risk neutral modeling in Section 2, we present a general approach to modeling of an emission market with one compliance period. More realistic multi-period models are treated in Section 5. The mathematical treatment of Section 3 is based on the analysis of diffusion martingales ending with only two possible values. We identify explicit classes of such martingales and we show how simple deterministic time changes can provide families of versatile risk neutral models for allowance prices. In Section 4, we demonstrate how to

Option Maturity	Option Type	Volume	Strike	Allowance Price	Implied Vol	Settlement Price
Dec-08	Cal	200,000	22.00	23.55	51.25%	5.06
Dec-08	Call	150,000	26.00	23.55	51.25%	3.57
Dec-08	Call	450,000	27.00	23.55	51.25%	3.27
Dec-08	Call	100,000	28.00	23.55	51.25%	2.99
Dec-08	Call	125,000	29.00	23.55	51.25%	2.74
Dec-08	Call	525,000	30.00	23.55	51.25%	2.51
Dec-08	Call	250,000	40.00	23.55	51.25%	1.04
Dec-08	Call	700,000	50.00	23.55	51.25%	0.45
Dec-08	Put	1,000,000	14.00	23.55	51.25%	0.64
Dec-08	Put	200,000	15.00	23.55	51.25%	0.86
Dec-08	Put	200,000	15.00	23.55	51.25%	0.86
Dec-08	Put	400,000	16.00	23.55	51.25%	1.13
Dec-08	Put	100,000	17.00	23.55	51.25%	1.43
Dec-08	Put	1,000,000	18.00	23.55	51.25%	1.78
Dec-08	Put	500,000	20.00	23.55	51.25%	2.60
Dec-08	Put	200,000	21.00	23.55	51.25%	3.07
Dec-08	Put	200,000	22.00	23.55	51.25%	3.57

Table 1: ECX EUA option quotes (in Euros) on January 4, 2008.

calibrate one of these models to historical allowance price data. We develop an historical calibration procedure because the option market has not yet matured to a point we can trust more standard calibration procedures based on option price data. The second part of the paper generalizes the one-period set-up to more realistic multi-periods models incorporating important features of real world markets, and Section 5 provides the necessary changes needed to extend the pricing formula to this more general set-up.

As a motivation for our derivations of option pricing formulas, we close this introduction with a short discussion of the idiosyncrasies of the EUA option markets. The facts reported below were a determining factor in our decision to write the present paper. European call and put options are actively traded on EUA futures contracts. Since 2006, trades of options maturing in December of each year (prior to 2012) have produced a term structure of option prices. On any given day, the volume varies from 5 to 25 million tons of CO_2-equivalent the short end of the curve being the most active with a good number of financial institutions involved, while the long end depends mostly on a few energy companies. It is not clear how these options are priced and a persistent rumor claims that traders plainly use Black's formula. The data reproduced in Table 1 is an extract of quotes published on January 4, 2008. Obviously, the implied volatility is perfectly flat, and the absence of skew or smile is consistent with the rumor. Whether or not traders are using Black or Black-Scholes formulas to price options on EAUs and futures contracts, we find it important to have option price formulas based on underlying martingales with binary terminal value, since Black-Scholes formula is based on an underlying price martingale converging to 0!

2 Risk Neutral Modeling of Emission Markets

In order to position our contribution within the existing literature, we briefly review the different methodologies of quantitative financial modeling.

Econometric Approach. It aims to give a description of statistical aspects in price movements. Thereby, the concrete nature of the underlying economic phenomena could be of secondary importance.

Equilibrium Approach. It focuses on the mechanics of price formation: given incentives, strategies, uncertainty, and risk aversions, the market is described by the cumulative effect of individual actions. Thereby, investigations are targeted on the understanding of market specifics, which is reached in a steady realistic state, the so-called market equilibrium.

Risk neutral approach It abstracts from the mechanism driving the market to the equilibrium state and focuses on its basic properties. Starting from the absence of arbitrage, the asset price evolution is introduced directly. This approach arose from the idea that although price movement is stochastic by its very nature, derivative valuation does not refer to the real-world probability. The description of statistical issues is not a goal of risk neutral models.

Absence of arbitrage is central to the risk neutral approach.. However, other equilibrium attributes may also be incorporated into the model. In the area of emission related financial assets, we have three basic insights from equilibrium modeling which are potentially important when establishing a risk neutral approach. The analysis of equilibrium for single compliance period schemes with penalty can be found in [6], It yields the following insights:

a) There is no arbitrage from trading allowances.

b) There are merely two final outcomes for the price of an allowance. Either the terminal allowance price drops to zero or it approaches the penalty level. Indeed, the price must vanish at maturity if there is excess of allowances, whereas in the case of shortage, the price will raise, reaching the level of the penalty. It is reasonable to suppose that in reality, the demand for allowances will coincidence with the supply with zero probability. So the occurrence of such an event will be disregarded.

c) Allowance trading instantaneously triggers all abatement measures whose costs are below the allowance price. The reason is that if an agent owns a technology with lower reduction costs than the present allowance price, then it is optimal to immediately use it to reduce his or her own pollution and profit from selling allowances.

At this point, we distinguish between two types of risk neutral approaches:

- the *reduced-form* risk neutral approach which focuses on a) and b).
- the *detailed* risk neutral approach which aims at all three properties a), b), and c).

Let us explain, at least at a formal level, the main differences between these classes of models. For the sake of concreteness, we focus on a continuous-time framework in which the risk-neutral evolution $(A_t)_{t\in[0,T]}$ of a futures contract written on terminal allowance price at compliance date T is given. In this framework, the allowance price process $(A_t)_{t\in[0,T]}$ is realized a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$, equipped with a distinct measure $\mathbb{Q} \sim \mathbb{P}$, which is interpreted as the *spot martingale measure*.

The *reduced-form* risk neutral approach focuses on the following problem

$$\boxed{\begin{array}{c}\text{Model the non-compliance event } N \in \mathcal{F}_T,\\ \text{which defines the } \mathbb{Q}\text{-martingale } (A_t)_{t\in[0,T]}\\ \text{with terminal value } A_T = \pi 1_N\end{array}} \tag{1}$$

The non-compliance event N is the only object that needs to be described exogenously. However to obtain a useful model, several requirements, ranging from computational tractability in derivatives valuation to diverse aspects of calibration must be fulfilled.

Within the *detailed* risk neutral approach, the non-compliance event N is obtained endogenously, in terms of other quantities which in turn, must be specified exogenously. This is where the issue c) comes into play. Under natural equilibrium assumptions (see [6]), the abatement activity in the market is driven by the allowance price in the following way: At any time t, given the allowance price A_t, the market exercises exactly those abatement measures whose costs are less than or equal to the market price A_t. This is also known as *the equilibrium allowance price equals to the marginal abatement costs* in environmental economics. Hence in equilibrium, the total abatement in the market can be described in terms of allowance prices $(A_s)_{s\in[0,T]}$ as

$$\int_0^T c_s(A_s)ds$$

where $c_s(a)(\omega)$ stands for the total intensity of the abatement measures at time $s \in [0, T]$ available in the market at price less than or equal to $a \in [0, \infty)$ in the market scenario $\omega \in \Omega$. In this context, the abatement volume function $c_t : [0, \pi] \times \Omega \to [0, \infty)$, $t \in [0, T]$ must be specified exogenously. In practice, the abatement volume function can be be estimated from market data: given a risk-neutral fuel price model, c_t can be described by an appropriate $\mathcal{B}[0, \pi] \otimes \mathcal{F}_t$-measurable functions, for each $t \in [0, T]$. In this context, the non-compliance event is given by

$$\{\omega \in \Omega;\ \mathcal{E}_T(\omega) - \int_0^T c(A_s)(\omega)ds \geq 0\}.$$

where an exogenously specified $\mathcal{F}_T$–measurable allowance demand $\mathcal{E}_T$ is used for the number of excess pollution units in the business-as-usual scenario (i.e. given zero penalty). Thus, the *detailed* risk neutral approach leads to a more complex mathematical problem

$$\boxed{\begin{array}{c}\text{Determine } (c_t)_{t\in[0,T]} \text{ from market data}\\ \text{and model allowance demand } \mathcal{E}_T \text{ to}\\ \text{obtain a } \mathbb{Q}\text{-martingale } (A_t)_{t\in[0,T]} \text{ with the}\\ \text{terminal value } A_T = \pi 1_{\{\mathcal{E}_T - \int_0^T c_t(A_t)dt \geq 0\}}.\end{array}} \tag{2}$$

Although the *detailed* risk neutral approach seems appealing from a methodological perspective, it is not obvious whether its higher complexity is justified from the viewpoint of derivative valuation. The present authors believe and show in this work that the *reduced-form* risk neutral approach yields satisfactory results, at least in the area of pricing plain-vanilla European options written on allowance futures. As illustration we investigate the solution of a particular problem of the type (2) and compare its results to a class of solutions to (1) suggested in the present work.

The existence and uniqueness of $(A_t)_{t\in[0,T]}$ solving (2) requires a delicate discussion. The martingale $\mathcal{E}_t = \mathbb{E}^{\mathbb{Q}}(\mathcal{E}_T \,|\, \mathcal{F}_t)$ defined for $t \in [0,T]$ plays an essential role. Indeed, the analysis of the discrete-time framework shows that if the future increments of A_t are independent of the present information at any time, then a solution to (2) should be expected in the functional form $A_t := \alpha(t, G_t)$ with an appropriate deterministic function $\alpha : [0,T] \times \mathbb{R} \ni (t,g) \mapsto \alpha(t,g) \in \mathbb{R}$ and a state process $(G_t)_{t\in[0,T]}$ given by

$$G_t = \mathcal{E}_t - \int_0^t c_s(A_s)ds, \qquad t \in [0,T].$$

This insight helps guess a solution in the standard diffusion framework, when there exists a process $(W_t, \mathcal{F}_t)_{t\in[0,T]}$ of Brownian motion with respect to $\mathbb{Q} \sim \mathbb{P}$, in the simplest case $d\mathcal{E}_t = \sigma dW_t$ with pre-specified $\sigma \in (0,\infty)$ and continuous, non-decreasing and deterministic abatement function $c : (0,\infty) \to \mathbb{R}$. Under these conditions, Itô's formula, applied to the martingale $A_t = \alpha(t, G_t)$ leads to a non-linear partial differential equation for α on $[0,T) \times \mathbb{R}$

$$\partial_{(1,0)}\alpha(t,g) - \partial_{(0,1)}\alpha(t,g)c(\alpha(t,g)) + \frac{1}{2}\partial_{(0,2)}\alpha(t,g)\sigma^2 = 0 \tag{3}$$

subject to the boundary condition

$$\alpha(T,g) = \pi 1_{[0,\infty)}(g) \quad \text{for all} \quad g \in \mathbb{R} \tag{4}$$

justified by the digital nature of the terminal allowance price. Having obtained α in this way, one constructs the state process $(G_t)_{t\in[0,T]}$ as a solution of the stochastic differential equation

$$dG_t = d\mathcal{E}_t - c(\alpha(t,G_t))dt, \qquad G_0 = \mathcal{E}_0 \tag{5}$$

from which we get a solution to (2) from $A_t = \alpha(t, G_t)$. Once $(A_t)_{t\in[0,T]}$ is determined, one applies standard integration to value European options. Although closed-form expressions are rare, option prices can be calculated numerically. The only case which yields quasi-explicit expressions (involving only numerical integrations) is that of linear abatement functions (see [20], [27]).

Let us elaborate on this case, to give the reader a feeling of allowance option pricing in the framework of *detailed* risk-neutral modeling.

Example Set the time to compliance date T to 2 years and assume the diffusion coefficient σ is 4, the penalty π 100, and suppose that the abatement function $c = c_t$ for $t \in [0,T]$ is linear with $c : a \mapsto 0.02 \cdot a$. At time $t = 0$, we consider a family of European calls with the

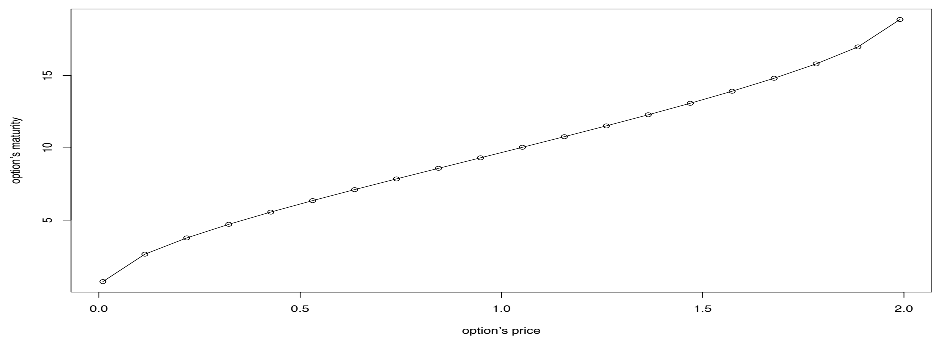

Figure 1: The price of a European call option as a function of its maturity.

same strike price of $K = 25$ but different maturity times τ running through $[0, T]$. Suppose that the initial allowance price equals to the strike price $a = A_0 = 25$. Determine call prices $C_0(\tau)$ at $t = 0$ for different maturity times $\tau \in [0, T]$. Independently of the model, the price of the Call in front of expiry date $\tau = 0$ must be equal to zero $C_0(0) = 0$, whereas the longest-maturity call $\tau = T$ must have a price $C_0(T) = A_0(\pi - K)/\pi = 25 \cdot 0.75 = 18.75$. Because of the digital terminal value of the underlying, such a Call is equivalent to 0.75 allowances. Call prices must increase with call's maturity from 0 to 18.75. This must be true *within any* risk neutral model. Figure 1 illustrates the exact curve $(C_0(\tau))_{\tau \in [0,T]}$ for the parameters as above. Since the end points are model-independent, merely intermediate-maturity prices exhibit model-dependent properties. Here we observe one remarkable issue: the so-called inverse S-shape.

In this work, we show that these features are shared by significantly simpler option pricing schemes (see Figure 4.2) based on the *reduced form* approach. For this reason, we believe that the reduced form approach can provide a reasonable pricing mechanism for emission-related financial products. However, we also agree that further development of detailed risk neutral, econometric, and equilibrium modeling is needed to help understand allowance price evolution. Certainly, suche models could be better suited to address the impact of information asymmetry, jumps in the information flow, regulatory uncertainty, and market idiosyncracies.

3 Reduced-Form Model for a Single Compliance Period

In this section, we introduce a simple model for an abstract emission market. We first restrict ourselves to a single compliance period, say $[0, T]$. The more realistic case of a multi-period models is treated in Section 5.

In the one-period setting, credits are allocated at the beginning of the period in order to enable allowance trading until time T and to encourage agents to exercise efficient abatement strategies. At the compliance date T, market participants cover their emissions by redeeming allowances, or pay a penalty π per unit of pollution not offset by credits. In this one-period model, unused allowances expire worthless as we do not allow for banking into the next period. Under natural assumptions, equilibrium analysis shows that the allowance price A_T at compliance date T is a random variable taking only the values 0 and π (see [6] and [5]). More precisely, if the market remains under the target pollution level, then the price approaches 0. Otherwise, the allowance price tends to the penalty level π.

All the relevant asset price evolutions are assumed to be given by adapted stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$, on which we fix an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ which we call the spot martingale measure.

We denote by $(A_t)_{t\in[0,T]}$ the price process of a future contract with maturity date T written on the allowance price. Given the digital nature of the terminal allowance price A_T, the central object of our study is the event $N \subset \mathcal{F}_T$ of non-compliance which settles the $\{0, \pi\}$-dichotomy of the terminal futures price by $A_T = \pi 1_N$. Furthermore, a standard no-arbitrage argument shows that the futures price $(A_t)_{t\in[0,T]}$ needs to be a martingale for the spot martingale measure $\mathbb{Q}$. Hence, the problem of allowance price modeling reduces to the appropriate choice of the martingale

$$A_t = \pi \mathbb{E}^{\mathbb{Q}}(1_N \,|\mathcal{F}_t), \quad t \in [0, T].$$

There are many candidates for such a process, but no obvious choice seems to be versatile enough for the practical requirements described below. An important requirement is the need to match the observed volatility structure. For a practitioner trying to calibrate at time $\tau \in [0, T]$ a model for the martingale $(A_s)_{s\in[\tau,T]}$ which finishes at 0 or π, the minimum requirements are to match the price observed at time τ, as well as the observed price fluctuation intensity up to this time τ. Further model requirements include the existence of closed-form formulas, or at least fast valuation schemes for European options, a small number of parameters providing sufficient model flexibility, and reliable and fast parameter identification from historical data. The goal of this paper is to present and analyze simple models satisfying these requirements.

In accordance with our earlier discussion of the two reduced form approaches, we choose our starting point to be the non-compliance event $N \in \mathcal{F}_T$ which we describe as the event where a hypothetic positive-valued random variable Γ_T exceeds the boundary 1, say $N = \{\Gamma_T \geq 1\}$. If one denotes by E_T the total pollution within the period $[0, T]$ which must be balanced against the total number $\gamma \in (0, \infty)$ of credits issued by the regulator, then the event of non-compliance should be given by $N = \{E_T \geq \gamma\}$ which suggests that Γ_T should be viewed as the normalized total emission E_T/γ. However, in our modeling, we merely describe the non-compliance event. Strictly speaking, so any random variable Γ_T with

$$\{\Gamma_T \geq 1\} = \{E_T/\gamma \geq 1\}$$

would do as well. On this account, we do not claim that Γ_T represents the total normalized emission E_T/γ. So the allowance spot price is given by the martingale

$$A_t = \pi \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T \geq 1\}} \,|\mathcal{F}_t), \quad t \in [0, T]$$

where the random variable Γ_T is chosen from a suitable parameterized family of random variables $\{\Gamma_T^\theta \,:\, \theta \in \Theta\}$. For reasons of model tractability, we suppose that the filtered probability space supports a process $(W_t)_{t\in[0,T]}$ of Brownian motion with respect to the spot martingale measure $\mathbb{Q}$, and we investigate parametric families which give allowance prices

$$A_t^\theta = \pi \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T^\theta \geq 1\}} \,|\mathcal{F}_t), \quad t \in [0,T]$$

with a Markovian stochastic evolution of the form

$$dA_t^\theta = v^\theta(t, A_t^\theta) dW_t$$

where the diffusion term v^θ captures the basic properties of historical price observations. In particular, we will match exactly the observed initial allowance price and the initial instantaneous price fluctuation intensity.

Remark We propose a consistent pricing scheme for emission related financial instruments within the framework of diffusion processes. Although this rules out discontinuity in allowance prices, we believe that this approach is reasonable. It has been argued that, due to jumps in the information flow, sudden allowance price changes must be included. However, based on our experience in the energy sector, possible allowance price jumps are not likely to play a significant role in mature emission markets. An increasing number of consultancies and market analyst is carefully watching the European emission market. Several agencies are providing news and periodical publications. Moreover, since energy generation and consumption are publicity observable, one should not expect significant allowance price jumps in a mature emissions market. On this account, a risk neutral model based on continuous allowance price evolution is reasonable.

To simplify the notation, we consider the normalized futures price process

$$a_t := \frac{1}{\pi} A_t = \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T \geq 1\}} \,|\, \mathcal{F}_t) \qquad t \in [0,T].$$

and we describe it under special assumptions on Γ_T. Our goal is to identify classes of martingales $\{a_t\}_{t\in[0,T]}$ taking values in the interval $(0,1)$, and satisfying

$$\mathbb{P}\{\lim_{t\nearrow T} a_t \in \{0,1\}\} = 1. \tag{6}$$

We first identify a parametric family of such martingales by working backward from a simple model for the random variable Γ_T motivated by intuitive understanding of the final cumulative level of emissions.

3.1 Basic Modeling of the Compliance Event

We use the discussion of the previous subsection as a motivation for the introduction of a compliance event of a specific form, and derive from there, the basic model whose theoretical properties and practical implementation will be discussed in the rest of the paper. We use the notation $N(\mu, \sigma^2)$ for the normal distribution with mean μ and variance σ^2, and write Φ for the cumulative distribution function of the standard normal distribution.

Proposition 1. *Suppose that*

$$\Gamma_T = \Gamma_0 e^{\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2 ds}, \quad \Gamma_0 \in (0,\infty) \tag{7}$$

for some continuous and square-integrable deterministic function $(0,T) \ni t \hookrightarrow \sigma_t$. *Then the martingale*

$$a_t = \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T \geq 1\}} \,|\, \mathcal{F}_t) \quad t \in [0,T] \tag{8}$$

is given by

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T \sigma_s^2 ds} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \tag{9}$$

and it solves the stochastic differential equation

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{z_t}dW_t \tag{10}$$

where the positive-valued function $(0,T) \ni t \hookrightarrow z_t$ *is given by*

$$z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}, \qquad t \in (0,T). \tag{11}$$

Remark. Notice that, even though the distribution of Γ_T depends only upon $\int_0^T \sigma_s^2 ds$, the non-compliance event N depends upon the entire function $\{\sigma_s\}_s$.

Proof. A direct calculation shows

$$\begin{aligned}
a_t &= \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T \geq 1\}} \,|\, \mathcal{F}_t) = \mathbb{Q}\{\Gamma_T \geq 1 \,|\, \mathcal{F}_t\} \\
&= \mathbb{Q}\left\{\Gamma_t e^{\int_t^T \sigma_s dW_s - \frac{1}{2}\int_t^T \sigma_s^2 ds} \geq 1 \,|\, \mathcal{F}_t\right\} \\
&= \Phi\left(\frac{\ln \Gamma_t - \frac{1}{2}\int_t^T \sigma_s^2 ds}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \\
&= \Phi\left(\frac{\ln(\Gamma_0) - \frac{1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}} \frac{\sqrt{\int_0^T \sigma_s^2 ds}}{\sqrt{\int_t^T \sigma_s^2 ds}} + \frac{\int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right),
\end{aligned}$$

and taking into account the initial condition

$$a_0 = \Phi\left(\frac{\ln \Gamma_0 - \frac{1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}}\right),$$

we obtain the desired expression (9). In order to show (10), we start with $a_t = \Phi(\xi_t), \quad t \in [0,T]$ where

$$\xi_t = \frac{\xi_{0,T} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}} \qquad \text{for } t \in [0,T], \quad \text{with} \quad \xi_{0,T} = \ln \Gamma_0 - \frac{1}{2}\int_0^T \sigma_s^2 ds \tag{12}$$

and $\xi_0 = \Phi^{-1}(a_0)$ with deterministic $a_0 \in (0,1)$. Computing its Itô differential, we get

$$\begin{aligned}
d\xi_t &= \left(\int_t^T \sigma_s^2 ds\right)^{-1/2} \sigma_t dW_t + \frac{1}{2}\left(x_0 + \int_0^t \sigma_s dW_s\right)\left(\int_t^T \sigma_s^2 ds\right)^{-3/2} \sigma_t^2 dt \\
&= \left(\int_t^T \sigma_s^2 ds\right)^{-1/2} \sigma_t dW_t + \frac{1}{2}\xi_t \left(\int_t^T \sigma_s^2 ds\right)^{-1} \sigma_t^2 dt \\
&= \sqrt{z_t} dW_t + \frac{1}{2} z_t \xi_t dt \\
d[\xi]_t &= z_t dt.
\end{aligned}$$

Next, Itô's formula gives the differential of the normalized allowance prices as

$$\begin{aligned}
da_t &= \Phi'(\xi_t) d\xi_t + \frac{1}{2}\Phi''(\xi_t) d[X]_t \\
&= \Phi'(\xi_t)\left(\sqrt{z_t} dW_t + \frac{1}{2} z_t \xi_t dt\right) + \frac{1}{2}\Phi''(\xi_t) z_t dt \\
&= \Phi'(\Phi^{-1}(a_t))\sqrt{z_t} dW_t
\end{aligned}$$

because $x\Phi'(x) + \Phi''(x) \equiv 0$. □

We notice for later use that if $t < \tau$, ξ_τ is given explicitly as a function of ξ_t by:

$$\xi_\tau = e^{\frac{1}{2}\int_t^\tau z_s ds} \xi_t + \int_t^\tau e^{\frac{1}{2}\int_s^\tau z_u du} \sqrt{z_s} dW_s. \tag{13}$$

3.2 Construction via Time Change

The stochastic differential equation (10) can be interpreted in the following way. Because of the factor $\sqrt{z_t}$ in front of dW_t, a_t can be viewed as the time-change of a martingale $\{Y_t\}_{t \in [0,\infty)}$ given by the strong solution of the stochastic differential equation:

$$dY_t = \Phi'(\Phi^{-1}(Y_t)) dW_t, \tag{14}$$

for $t \in [0,\infty)$, with $Y_0 \in (0,1)$. This solution stays in the open interval $(0,1)$ and converges to the boundaries 0 or 1 with certainty when t approaches ∞.

$$\mathbb{P}\{\lim_{t \nearrow \infty} Y_t \in \{0,1\}\} = 1. \tag{15}$$

This construction is in fact a special case of a general program where the martingale $\{a_t\}_{t \in [0,T]}$ satisfying (22) is constructed in two steps: first determine a $(0,1)$-valued martingale $\{Y_t\}_{t \in [0,\infty)}$ satisfying (15), and then search for a time change bringing the half-axis $[0,\infty)$ onto the bounded interval $[0,T)$. With this in mind, it appears natural to consider the solutions of the stochastic differential equation

$$dY_t = \Theta(Y_t) dW_t, \quad Y_0 \in (0,1), \qquad t \geq 0, \tag{16}$$

where Θ is a nonnegative continuous function on $[0,1]$ satisfying $\Theta(0) = \Theta(1) = 0$. We can then use Feller's classification (see for example [12] or [13]) to check that such a diffusion is conservative, does not reach the boundaries 0 and 1 in finite time, and satisfies (15). This is the case if $v(0+) = v(1-) = \infty$ where $v(x)$ is defined by

$$v(x) = 2\int_{0.5}^{x}(x-y)\frac{dy}{\Theta(y)}, \qquad x \in (0,1).$$

Straightforward computations show that the solution of the stochastic differential equation (14) does indeed satisfy these conditions, hence it does not hit 0 and 1 in finite time with probability one.

Explicit families of such martingales can easily be constructed. Case in point, a two-parameter family of examples can be constructed from the analysis of [3] which we learned from Mike Terhanchi (who extended the argument of [3] to Lévy processes in [23]). If we set

$$X_t = e^{-W_t+ct}\left(X_0 - \int_0^t e^{W_s-cs}(a\,ds + dB_s)\right), \qquad X_0 \in \mathbb{R}$$

for $c > 0$ and $a \in \mathbb{R}$ where $\{W_t\}_{t\in[0,\infty)}$ and $\{B_t\}_{t\in[0,\infty)}$ are independent Wiener processes, then $\{X_t\}_{t\in[0,\infty)}$ satisfies

$$dX_t = \left[\left(c+\frac{1}{2}\right)X_t - a\right]dt\ - X_t dW_t - dB_t, \qquad t \in [0,\infty)$$

and

$$\lim_{t\to\infty} X_t = -\infty \qquad \text{on the set } \{\int_0^\infty e^{W_s-cs}(a\,ds+dB_s) > X_0\},$$

$$\lim_{t\to\infty} X_t = +\infty \qquad \text{on the set } \{\int_0^\infty e^{W_s-cs}(a\,ds+dB_s) < X_0\}.$$

Now if we define the function G by $G(x) = \int_{-\infty}^{x} g(y)dy$ for all $x \in \mathbb{R}$ where the function g is

$$g(y) = C\frac{e^{2a\tan^{-1}y}}{(1+y^2)^{c+1/2}}, \qquad y \in \mathbb{R},$$

with the constant $C > 0$ chosen so that $\int_{-\infty}^{+\infty} g(y)dy = 1$, then it is easy to check that

$$\frac{1}{2}g'(y)(1+y^2) + g(y)[(c+\frac{1}{2})y - a] = 0$$

which in turn implies that $Y_t = G(X_t)$ is a martingale. Clearly, this martingale satisfies the limits (15). Moreover, a simple application of Itô's formula shows that $\{Y_t\}_t$ is a solution of the stochastic differential equation (16) with $\Theta(y) = g(G^{-1}(y))\sqrt{1+G^{-1}(y)^2}$.

It is now plain to see that the basic model of Proposition 1 is a particular case of this construction. Indeed, if $(Y_t = \Phi(X_t))_{t\in[0,\infty[}$ for

$$X_t = e^t\big(x_0 + \int_0^t e^{-s}dW_s\big), \qquad x_0 \in \mathbb{R}, \qquad \text{for all } t \in [0,\infty).$$

and $(z_s)_{s\in[0,T)}$ is a positive-valued, continuous function , then the $(0,1)$-valued process

$$a_t = Y_{\int_0^t z_s ds}, \qquad t \in [0,T[,$$

satisfies

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{z_t}d\tilde{W}_t, \qquad t \in [0,T[$$

for the Brownian motion $(\tilde{W}_t, \tilde{\mathcal{F}}_t)_{t\in[0,T[}$

$$\tilde{W}_t = \sqrt{\frac{t}{\int_0^t z_s ds}} W_{\int_0^t z_s ds} \qquad \text{with} \qquad \tilde{\mathcal{F}}_t = W_{\int_0^t z_s ds}.$$

Moreover,

$$\lim_{t\to T}\int_0^t z_s ds = +\infty \quad \Longrightarrow \quad \mathbb{P}\{\lim_{t\nearrow T} a_t \in \{0,1\}\} = 1.$$

4 Model Parametrization and Calibration

We now show how to calibrate the basic model introduced in Section 3.1. Option data were not available at the time this paper was written. As a consequence we limit ourselves to historical calibration of the model.Note that according to Lemma **??**, the choice of the function $(0,T) \ni t \hookrightarrow \sigma_s$ affects only the time-change $(z_t)_{t\in(0,T)}$. Moreover, Proposition 1 shows that when modeling the random variable Γ_T by (7), we must assume that the function $(0,T) \ni t \hookrightarrow \sigma_s$ is not constant. Indeed, a constant volatility

$$\sigma_s \equiv \bar{\sigma} \in (0,\infty) \quad \text{for all} \;\; s \in [0,T]$$

would give, independently on the choice of $\bar{\sigma}$, the same process

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{T} + \; W_t}{\sqrt{T-t}}\right) \tag{17}$$

with dynamics

$$da_t = \Phi'(\Phi^{-1}(a_t))\frac{1}{\sqrt{T-t}}dW_t. \tag{18}$$

Thus, with a constant and deterministic $\bar{\sigma}$ it is impossible to match both, the recent allowance price and the recently observed (instantaneous) fluctuation intensity. Indeed, the entire process is completely determined by the value of a_0. This suggests that we introduce extra degrees of freedom in (18). In this paper we choose to work with the model

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{\beta(T-t)^{-\alpha}}dW_t \tag{19}$$

parameterized by $\alpha \in \mathbb{R}$ and $\beta \in (0,\infty)$. This leads to a parametric family of functions $(\sigma_s)_{s\in[0,T]}$ which we denote by

$$(\sigma_s(\alpha,\beta))_{s\in(0,T)}, \qquad \alpha \geq 1, \beta > 0, \tag{20}$$

and we show how to calibrate the parameterized family (20) to historical data.

As seen from (7), the function $(0,T) \ni s \hookrightarrow \sigma_s$ enters the dynamics of $(a_t)_{t\in[0,T]}$ indirectly through the time-change function $(0,T) \ni t \hookrightarrow z_t$ defined in (11). The correspondence between the functions σ and z is elucidated in the following lemma.

Lemma 1. *a) Given any square-integrable continuous and positive function* $(0,T) \ni s \hookrightarrow \sigma_s$*, the function* $(0,T) \ni t \hookrightarrow z_t$ *defined by*

$$z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}, \qquad t \in (0,T), \tag{21}$$

is positive, continuous and satisfies

$$\lim_{t \nearrow T} \int_0^t z_u du = +\infty. \tag{22}$$

b) Conversely, if the positive and continuous function $(0,T) \ni t \hookrightarrow z_t$ *satisfies (22) then the function* $(0,T) \ni s \hookrightarrow \sigma_s$ *defined by*

$$\sigma_t = \sqrt{z_t e^{-\int_0^t z_u du}}, \qquad t \in (0,T).$$

is positive, continuous and satisfies (21).

Proof. a) Let us write (21) as $z_t \varphi_t = \sigma_t^2$ for $t \in (0,T)$ where

$$\varphi_t = \int_t^T \sigma_u^2 du \quad \text{for all } t \in [0,T].$$

Then, $\dot{\varphi}_t = -\sigma_t^2$ for $t \in (0,T)$ and φ satisfies the differential equation $z_t \varphi_t = -\dot{\varphi}_t$ for $t \in (0,T)$. Its solution is given by

$$\varphi_t = \varphi_0 e^{-\int_0^t z_u du} \qquad t \in [0,T).$$

Form the terminal condition $\varphi_T = \int_T^T \sigma_u^2 du = 0$, we get (22).

b) Let us now suppose that $(z_t)_{t\in(0,T)}$ is positive, continuous and satisfies (22), and let us define the positive and continuous function $(\varphi_t)_{t\in[0,T)}$ by

$$\varphi_t = e^{-\int_0^t z_u du} \qquad t \in [0,T).. \tag{23}$$

Clearly, it satisfies $\dot{\varphi}_t = -z_t \varphi_t$ for $t \in (0,T)$, and since the divergence of the integral implies that $\varphi_T = 0$, we have

$$\varphi_t = -\int_t^T \dot{\varphi}_u du \qquad t \in (0,T) \qquad \text{and} \qquad -z_t = -\frac{\dot{\varphi}_t}{\int_t^T \dot{\varphi}_u du}. \tag{24}$$

Setting $\sigma_t^2 = -\dot{\varphi}_t$ for $t \in (0,T)$, (21) is satisfied. Moreover, this function is positive, continuous in the open interval, and integrable since $1 = \varphi(0) = -\int_0^T \dot{\varphi}_u du$ which follows from (23) and (24). Consequently, the function defined by $\sigma_t := \sqrt{\sigma_t^2}$ for $t \in (0,T)$ is square integrable, continuous, positive and is related to $(z_t)_{t\in(0,T)}$ by (21), as required. □

We return to the expression (7) for Γ_T, using now the targeted family (20) to determine the stochastic differential equation (19). In light of the previous lemma, the function

$$(z_t(\alpha,\beta) = \beta(T-t)^{-\alpha})_{t\in(0,T)}, \tag{25}$$

must satisfy (22), implying the following restrictions on the parameters α and β:

$$\beta > 0 \quad \text{and} \quad \alpha \geq 1. \tag{26}$$

However, we will let $alpha$ and $beta$ vary freely over $\mathbb{R}$ for calibration purposes, interpreting the fitted values in light of these conditions.

Remark: If we use the parametric family $z_t(\alpha,\beta) = \beta(T-t)^{-\alpha}$, then the actual time change is given by the integral

$$\int_0^t z_s ds = \begin{cases} \beta(\log(T) - \ln(T-t)) & \text{if } \alpha = 1 \\ \frac{\beta}{1-\alpha}[T^{1-\alpha} - (T-t)^{1-\alpha}] & \text{otherwise} \end{cases}$$

Notice that β is a multiplicative parameter in the sense that $z_t(\alpha,\beta) = \beta z_t(\alpha,1)$. Also, the *emission volatility* $(\sigma_t(\alpha,\beta))_{t\in(0,T)}$ associated to the parameterization $(z_t(\alpha,\beta))_{t\in[0,T]}$ is given by:

$$\sigma_t(\alpha,\beta)^2 = z_t(\alpha,\beta)e^{-\int_0^t z_u(\alpha,\beta)du} \tag{27}$$

$$= \begin{cases} \beta(T-t)^{-\alpha}e^{-\frac{\beta}{1-\alpha}[T^{1-\alpha}-(T-t)^{1-\alpha}]} & \text{if } \alpha \neq 1 \\ \beta(T-t)^{\beta-1}T^{-\beta} & \text{if } \alpha = 1. \end{cases} \tag{28}$$

4.1 Historical Calibration

Consider historical observations of the futures prices $(A_t)_{t\in[0,T]}$, recorded at times $t_1 < t_2 <, \ldots, < t_n$. resulting in a data set $\xi_1, \cdots, \xi_n$ where

$$\xi_i = \Phi^{-1}(a_{t_i}) = \Phi^{-1}\left(\frac{1}{\pi}A_{t_i}\right), \qquad i = 1, \cdots, n. \tag{29}$$

The objective measure $\mathbb{P}$ governing the statistics of the observations can be recovered from the spot martingale measure $\mathbb{Q}$ via its Radon-Nikodym density

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{\int_0^T H_t dW_t - \frac{1}{2}\int_0^T H_t^2 dt}.$$

For the sake of simplicity, we follow the time-honored approach assuming that the market price of risk process $(H_t)_{t\in[0,T]}$ is constant and deterministic, $H_t \equiv h$ for $t \in [0,T]$, for some fixed $h \in \mathbb{R}$. According to Girsanov's theorem, the process $\{\tilde{W}_t\}_{t\in[0,T)}$ defined as $\tilde{W}_t = W_t - ht$ for $t \in [0,T)$ is a Brownian motion with respect to the objective measure $\mathbb{P}$, and under this measure $\mathbb{P}$, ξ_t satisfies:

$$d\xi_t = (\frac{1}{2}z_t\xi_t + h\sqrt{z_t})dt + \sqrt{z_t}d\tilde{W}_t$$

and the analog of (13) reads

$$\xi_\tau = e^{\frac{1}{2}\int_t^\tau z_s ds}\xi_t + h\int_t^\tau e^{\frac{1}{2}\int_s^\tau z_u du}\sqrt{z_s}ds + \int_t^\tau e^{\frac{1}{2}\int_s^\tau z_u du}\sqrt{z_s}dW_s. \tag{30}$$

for $0 \le t \le \tau \le T$. Consequently, for each $i = 1, \cdots, n$, the conditional distribution of ξ_i given ξ_{i-1} is Gaussian with mean μ_i and variance σ_i^2 given by

$$\mu_i(h,\alpha,\beta) = e^{\frac{1}{2}\int_{t_{i-1}}^{t_i} z_s ds}\xi_{t_{i-1}} + h\int_{t_{i-1}}^{t_i} e^{\frac{1}{2}\int_s^{t_i} z_u du}\sqrt{z_s}ds, \qquad \sigma_i^2(h,\alpha,\beta) = \int_{t_{i-1}}^{t_i} z_s e^{\int_s^{t_i} z_u du}ds,$$

provided we fix t_0 and ξ_0 by convention. Notice that $\sigma_i^2(h,\alpha,\beta) = \sigma_i^2(\alpha,\beta)$ does not depend upon h and that $\mu_i(h,\alpha,\beta)$ is an affine function of h in the sense that:

$$\mu_i(h,\alpha,\beta) = \mu_i^0(\alpha,\beta) + h\mu_i^1(\alpha,\beta)$$

and simple arithmetic gives

$$\mu_i^0(\alpha,\beta) = \begin{cases} \xi_{i-1}\left(\frac{T-t_{i-1}}{T-t_i}\right)^{\beta/2} & \text{if } \alpha = 1 \\ \xi_{i-1}\exp\left[\frac{\beta}{2(1-\alpha)}[(T-t_{i-1})^{1-\alpha} - (T-t_i)^{1-\alpha}]\right] & \text{if } \alpha \neq 1, \end{cases} \tag{31}$$

and

$$\mu_i^1(\alpha,\beta) = \begin{cases} \frac{2\sqrt{\beta}}{\beta+1}\sqrt{T-t_i}\left[\left(\frac{T-t_{i-1}}{T-t_i}\right)^{(\beta+1)/2} - 1\right] & \text{if } \alpha = 1 \\ \sqrt{\beta}e^{-\beta(T-t_i)^{1-\alpha}/(2(1-\alpha))}\int_{t_{i-1}}^{t_i}(T-s)^{-\alpha/2}e^{\beta(T-s)^{1-\alpha}/(2(1-\alpha))}ds & \text{if } \alpha \neq 1, \end{cases} \tag{32}$$

while the variance $\sigma_i^2(\alpha,\beta)$ is given by

$$\sigma_i^2(\alpha,\beta) = \begin{cases} \left(\frac{T-t_{i-1}}{T-t_i}\right)^{\beta} - 1 & \text{if } \alpha = 1 \\ \exp\left[\frac{\beta}{1-\alpha}[(T-t_{i-1})^{1-\alpha} - (T-t_i)^{1-\alpha}]\right] - 1 & \text{if } \alpha \neq 1. \end{cases} \tag{33}$$

So for a given realization $\{\xi_i\}_{i=1}^n \in \mathbb{R}^n$, the log-likelihood is

$$L_{\xi_1,\ldots,\xi_n}(h,\alpha,\beta) = \sum_{i=1}^n \left(-\frac{(\xi_i - \mu_i^0(\alpha,\beta) - h\mu_i^1(\alpha,\beta))^2}{2\sigma_i^2(\alpha,\beta)} - \log(\sqrt{2\pi\sigma_i^2(\alpha,\beta)})\right) \tag{34}$$

for all $h, \alpha, \beta \in \mathbb{R}$, and setting the partial derivative of $L_{\xi_1,\ldots,\xi_n}(h,\alpha,\beta)$ with respect to h to 0 gives:

$$h^* = \frac{\sum_{1\le i\le n}(\xi_i - \mu_i^0(\alpha,\beta))/\sigma_i^2(\alpha,\beta)}{\sum_{1\le i\le n}(\mu_i^1(\alpha,\beta))^2/\sigma_i^2(\alpha,\beta)}. \tag{35}$$

We can then substitute this value into (34) and optimize with respect to α and β.

THIS PART NEEDS TO BE UPDATED

For fixed $\alpha \ge 1$, the maximum-likelihood estimation of the parameters $h \in \mathbb{R}$ and $\beta \in (0,\infty)$ can be computed explicitly.

Lemma 2. *For fixed* $\alpha \geq 1$ *the maximum of* $(h, \beta) \to L_{y_1,\ldots,y_n}(h, \beta, \alpha)$ *on* $(h, \beta) \in \mathbb{R} \times (0, \infty)$ *is attained at the values* $(h^*, \beta^*) = (h^*(\alpha), \beta^*(\alpha))$ *given in terms of*

$$x^* = \left(\sum_{i=1}^n \frac{y_i}{\sqrt{z_i(\alpha)}} \right) \left(\sum_{i=1}^n \Delta_i \right)^{-1}$$

by

$$\beta^* = \frac{1}{n} \sum_{i=1}^n \frac{(y_i - \Delta_i \sqrt{z_i(\alpha)} x^*)^2}{\Delta_i z_i(\alpha)}, \tag{36}$$

$$h^* = \frac{1}{\sqrt{\beta^*}} x^* \tag{37}$$

Proof. For fixed $\alpha \geq 1$, fix $\beta \in (0, \infty)$ momentarily in order to compute $h(\beta, \alpha)$ as the maximizer of $h \mapsto L_{y_1,\ldots,y_n}(h, \beta, \alpha)$. It is obtained by solving

$$\frac{\partial}{\partial h} L_{y_1,\ldots,y_n}(h, \beta, \alpha) = \sum_{i=1}^n \frac{2(y_i - \sqrt{z_i(\alpha)}\sqrt{\beta} h \Delta_i)}{2\Delta_i \beta z_i(\alpha)} \sqrt{z_i(\alpha)}\sqrt{\beta}\Delta_i = 0.$$

which is equivalent to

$$\sum_{i=1}^n \frac{y_i}{\sqrt{z_i(\alpha)}\sqrt{\beta}} = \sum_{i=1}^n h\Delta_i \;\; \Rightarrow \;\; h(\beta) = \frac{1}{\sqrt{\beta}} \left(\sum_{i=1}^n \frac{y_i}{\sqrt{z_i(\alpha)}} \right) \left(\sum_{i=1}^n \Delta_i \right)^{-1}.$$

Now plug the expression

$$h(\beta, \alpha) = \frac{1}{\sqrt{\beta}} x^* \tag{38}$$

into the formula (34) of the likelihood density and find $\beta^*(\alpha)$ as the maximizer of $\beta \mapsto L_{y_1,\ldots,y_n}(h(\beta, \alpha), \beta, \alpha)$ on $\beta \in (0, \infty)$ by solving

$$\frac{\partial}{\partial \beta} L_{y_1,\ldots,y_n}(h(\beta, \alpha), \beta, \alpha) = \sum_{i=1}^n \left(\frac{(y_i - \sqrt{z_i(\alpha)}\Delta_i x^*)^2}{2\Delta_i \beta^2 z_i(\alpha)} - \frac{1}{2\beta} \right) = 0$$

which is equivalent to

$$\sum_{i=1}^n \frac{(y_i - \sqrt{z_i(\alpha)}\Delta_i x^*)^2}{\Delta_i z_i(\alpha)} = n\beta^*$$

and gives (36). Finally, β^* must be plugged into (38) to obtain (37). □

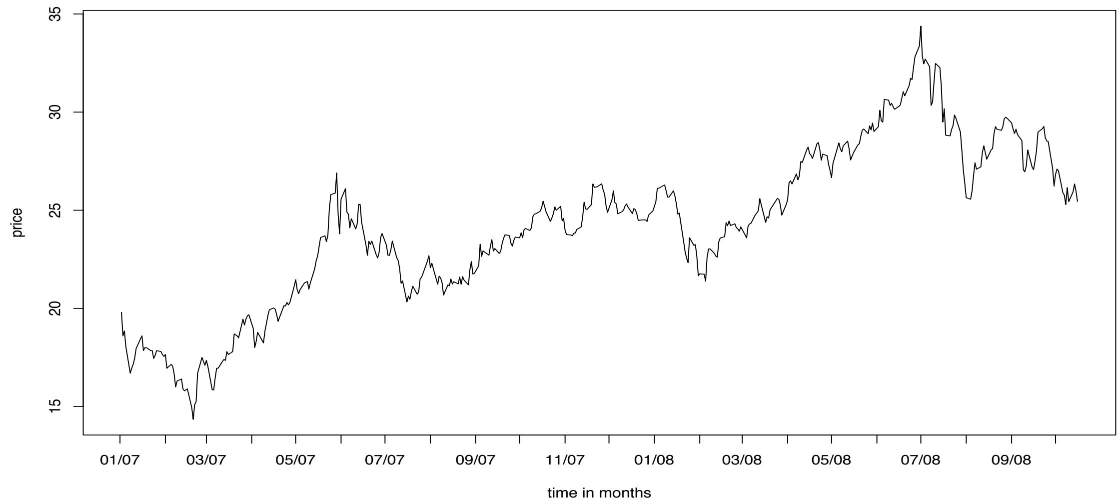

Figure 2: Future prices on EUA with maturity Dec. 2012

To illustrate this result, we set $\alpha = 1$ and computed the above estimates for the futures prices reproduced on Figure 2, of the European Union Allowance for the second phase of the European Emission Trading Scheme.

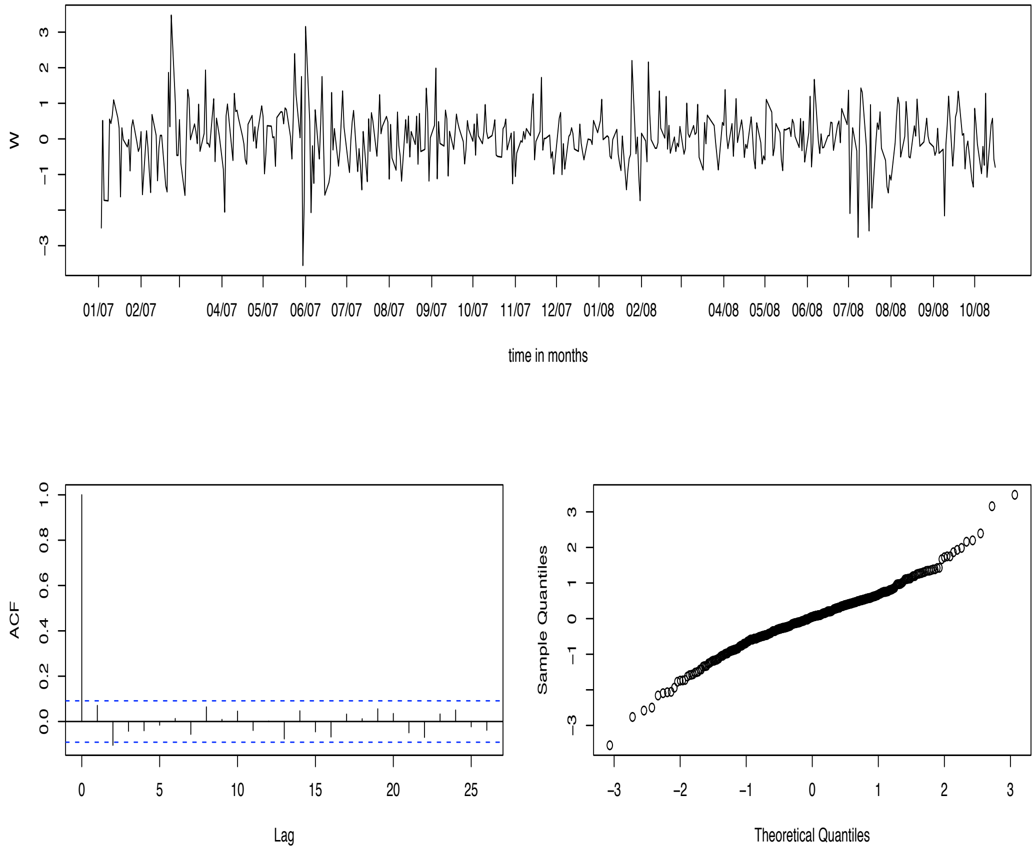

Figure 3: Statistical analysis of series $(w_i)_{i=1}^n$.

Normalizing these historical futures prices as in (29) with $\pi = 100$, we extracted a realization $(y_i)_{i=1}^n \in \mathbb{R}^n$ of the series $(Y_i)_{i=1}^n$ defined in (**??**). The estimations (37) produced the values

$$h^* = h^*(1) = 0.4656, \qquad \beta^* = \beta^*(1) = 0.4377.$$

To verify the validity of our procedure, we determine

$$w_i = \frac{y_i - \sqrt{z_i(1)}\sqrt{\beta^*}h^*\Delta_i}{\sqrt{z_i(1)\beta^*\Delta_i}}, \qquad i = 1, \ldots, n,$$

Standard statistical analysis of these *residuals* can be applied to check the goodness of thel fit. Namely, under the model assumptions, this series must be a realization of independent standard normal random variables. In Figure 3 we show this series, its empirical auto-correlation and its qq-plot. In line with noise-like auto-correlation, the Box-Ljung test shows no indication to reject the independence hypothesis giving the p-value 0.1275 at lag one. However, although qq-plot seems to be an almost straight line, the normality is rejected by both, the Jarque-Bera and the Shapiro-Wilk tests.

Remark The rejection of normality hypothesis and a relatively low p-value of the Box-Ljung test should not be interpreted as an indication of a poor model fit. The reader should keep in mind that risk neutral models are not designed to capture all statistical particularities of the underlying financial time series. In fact, their target is the price evolution with respect to an artificial (non-physical) risk neutral measure. For instance, the hypothesis on normality of the log-returns is rejected for almost all financial data, but is supposed by the celebrated Black-Scholes model for the risk neutral price movement. In our case, statistical arguments are based on the ad hoc connection between risk neutral and objective measure via constant and deterministic Girsanov kernel. Strictly speaking, any application of sophisticated statistical techniques is methodologically wrong in this setting. The only satisfactory way to fit the model to the data would be the implied calibration based on the identification of those model parameters, which best explain the listed derivatives prices. Due to limited number of derivatives, the implied calibration is impossible at the current market state. Hence, the historical calibration should be considered as tradeoff, which yields reasonable results, at least in the present study.

Finally, consider the identification of $\alpha \geq 1$. Although there is no closed-form estimate for this parameter, the maximum of the likelihood function can be determined numerically. After plugging in $(h^*(\alpha), \beta^*(\alpha))$ from Lemma 2 into the likelihood function, we need to find the maximizer α^* of $L_{y_1,\dots,y_n}(h^*(\alpha), \beta^*(\alpha), \alpha)$ over $\alpha \geq 1$. Figure 4 gives the plot of this likelihood as a function of α.

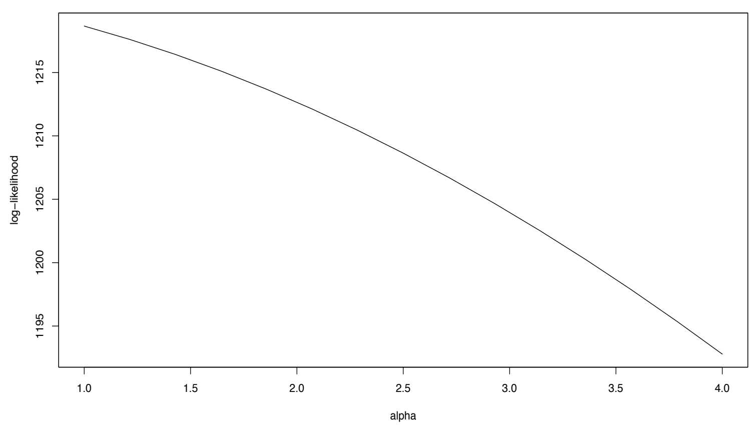

Figure 4: The maximum at $\alpha^* = 1$ of $\alpha \mapsto L_{y_1,\dots,y_n}(h^*(\alpha), \beta^*(\alpha), \alpha)$.

Obviously, $\alpha = 1$ is a local maximum. Apparently, the allowance price data from Figure 2 give no support to the assumption that $\alpha > 1$. On this account, we suppose for the reminder of this work that $\alpha = 1$. We work with the one-parameter family

$$\sigma_t(\beta)^2 = \beta(T-t)^{\beta-1}T^{\beta}, \qquad t \in (0,T), \qquad \beta > 0 \tag{39}$$

to describe the risk-neutral allowance price time-evolution, and address the problem of option pricing under this standing assumption.

END OF THE PART WHICH NEEDS TO BE UPDATED / SHORTENED

Remark The original rationale for the choice of our basic model was based on equilibrium considerations and the assumption that the terminal allowance price was binary. However, the real market EU ETS (second phase), whose data are used for historical calibration, operates under uncertainty. One of the major price determinant here is the unknown impact of the international credits, the so-called Certified Emission Reductions or CERs. Most likely, market participants believe that a significant number of cheap international credits will be used to fulfill the compliance within EU ETS if needed, and that non-compliance because of a shortage of certificates will not occur at compliance time. Under such condition, the distribution of the terminal allowance price should not be binary any more. Namely, in the case of national allowance shortage, it would reach a level determined by supply and demand for international credits, which is likely to fall below the EU ETS penalty of 100 EURO. It is interesting to see that historical data seem to reflect this concern, suggesting a value for the parameter α below 1 which would yield a martingale with a non-digital terminal value since the integral giving the time change does not diverge when $\alpha < 1$!

Remark. The above maximum likelihood calibration from historical data used strongly the explicit form (8) of the normalized allowance price and the Gaussian property of the diffusion ξ_t given by the explicit form of the solution given by (13). For general models of one dimensional diffusion processes with volatility given by a more function Θ more general than the specific $\Theta = \Phi' \circ \Phi^{-1}$, the maximum likelihood estimates of α, β and h can be computed using Ait-Sahalia's approach introduced in [1].

4.2 Option Pricing

Now, we turn our attention to the valuation of European call options written on allowance futures price $(A_t)_{t\in[0,T]}$. The payoff of a European call with maturity $\tau \in [0,T]$ and strike price $K \geq 0$ is given by $(A_\tau - K)^+$. Under the assumption that the savings account $\{B_t\}_{t\in[0,T]}$ is given by $B_t = e^{\int_0^t r_s B_s ds}$ for $t \in [0,T]$ for some deterministic short rate $\{r_s\}_{s\in[0,T]}$, this price can be computed in the model proposed in this paper.

Proposition 2. *In a one-period $[0,T]$ compliance model, with risk neutral parameters $\alpha \in \mathbb{R}$ and $\beta > 0$, the price of a European call with maturity $\tau \in [0,T]$ and strike $K \geq 0$ written on an allowance futures maturing at the end T of the compliance period is given at time $t \in [0,\tau]$ by*

$$C_t = e^{-\int_t^\tau r_s ds} \int_{\mathbb{R}} (\pi\Phi(x) - K)^+ N(\mu_{t,\tau}, \sigma^2_{t,\tau})(dx) \tag{40}$$

with $\mu_{t,\tau}$ and $\sigma^2_{t,\tau}$ given by formulas (41) and (43) below.

This result is obtained by a straightforward calculation using the fact which we already stressed several times that the conditional distribution of ξ_τ given ξ_t is Gaussian with mean

$$\mu_{t,\tau}(\alpha,\beta) \tag{41}$$

$$= \begin{cases} \xi_t \left(\frac{T-t}{T-\tau}\right)^{(\beta/2)} & \text{if } \alpha = 1 \\ \xi_t \exp\left[\frac{\beta}{2(1-\alpha)}[(T-t)^{1-\alpha} - (T-\tau)^{1-\alpha}]\right] & \text{if } \alpha \neq 1. \end{cases} \tag{42}$$

and variance

$$\sigma^2_{t,\tau}(\alpha,\beta) = \begin{cases} \left(\frac{T-t}{T-\tau}\right)^{\beta} - 1 & \text{if } \alpha = 1 \\ \exp\left[\frac{\beta}{1-\alpha}[(T-t)^{1-\alpha} - (T-\tau)^{1-\alpha}]\right] - 1 & \text{if } \alpha \neq 1. \end{cases} \tag{43}$$

Let us illustrate the role of the parameter β on option prices. In the following example, we assume that the penalty is $\pi = 100$ and we suppose that at the initial time $t = 0$ four years prior to the compliance date $T = 4$ the price of a futures contract, written on allowance price at T is $A_0 = 25$. For constant and deterministic continuously compounded interest rate $r = 0.05$ we consider European calls written on the forward price with strike price of $K = 25$ and varying maturity date $\tau \in [0, T]$. The option price is calculated from (40) at time $t = 0$. In Figure 4.2, we also illustrate the dependence of the option price upon the parameter β (recall Figure 1 for a plot for fixed β). Comparing three cases $\beta = 0.5$, $\beta = 0.8$ and $\beta = 1.1$, Figure 4.2 shows that the call price is increasing in β. Less surprisingly, the dependence on τ shows that longer-maturity calls (with the same strike) are more valuable than their short-maturity counterparts.

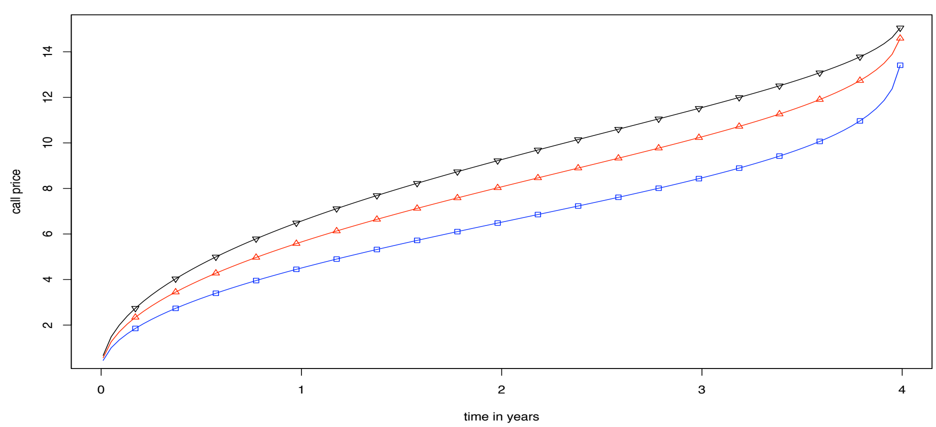

Figure 5: Plots of the prices $C_0(\tau)$ at time $t = 0$ as functions of option maturity τ. The graphs marked by □, △, and ▽ stand for $\beta = 0.5$, $\beta = 0.8$, and $\beta = 1.1$ respectively. The values of the other parameters are given in the text.

Let us stress the fact that, although there are no closed-form formulas for call prices, their numerical evaluations can be performed very efficiently.

5 Multi-Compliance Periods Markets

So far, we focused on a generic cap and trade scheme modeled after the first phase of the EU ETS, namely limited to one compliance period and without banking in the sense that unused allowances become worthless at the end of the period. This is a strong simplification since as already mentioned above, real-world markets are operating in a multi-period framework. Furthermore, subsequent periods are connected by market specific regulations. In what follows, we consider an abstract but generic model of such a market and focus on most natural rules for the period interconnection.

Presently, there are three regulatory mechanisms connecting successive compliance periods in a cap-and-trade scheme. Their rules go under the names of *borrowing*, *banking* and *withdrawal*.

- Borrowing allows for the transfer of a (limited) number of allowances from the next period into the present one;
- Banking allows for the transfer of a (limited) number of (unused) allowances from the present period into the next;
- Withdrawal penalizes firms which fail to comply in two ways: by penalty payment for each unit of pollutant which is not covered by credits and by withdrawal of the missing allowances from their allocation for the next period.

From the nature of the existing markets and the designs touted for possible implementation, it seems that policy makers tend to favor unlimited banking and forbid borrowing. Furthermore, the withdrawal rule is most likely to be included. Banking and withdrawal seem to be reasonable rules to reach an emission target within a fixed number of periods, because each success (resp. failure) in the previous period results in stronger (resp. weaker) abatement in the subsequent periods.

5.1 Market Model

For the remainder of this section, we consider a two-period market model without borrowing, and with withdrawal and unlimited banking. We denote the two periods by $[0, T]$ and $[T, T']$ and consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T']})$ with a distinct measure $\mathbb{Q} \sim \mathbb{P}$ which we view as the spot martingale measure. Further, we introduce processes $(A_t)_{t\in[0,T]}$, $({A'}_t)_{t\in[0,T']}$ for the futures contracts with maturities at compliance dates T, T' written on allowance prices from the first and the second period respectively. In order to exclude arbitrage, we suppose that the prices $(A_t)_{t\in[0,T]}$ and $(A'_t)_{t\in[0,T']}$ are martingales with respect to the spot martingale measure $\mathbb{Q}$. Non-compliance in the first and in the second periods occur on events $N \in \mathcal{F}_T$ and $N' \in \mathcal{F}_{T'}$ respectively. As before, we assume that the savings account $(B_t)_{t\in[0,T']}$ is given by

$$B_t = e^{\int_0^t r_s ds}, \qquad t \in [0, T'] \tag{44}$$

for some deterministic short rate $(r_s)_{s\in[0,T']}$. The results of the previous section imply that in the case $\Omega \setminus N$ of the first-period compliance the allowance price drops

$$A_T 1_{\Omega\setminus N} = \kappa A'_T 1_{\Omega\setminus N}, \tag{45}$$

where $\kappa \in (0, \infty)$ stands for discount factor describing the interest rate effect

$$\kappa = B_T B_{T'}^{-1} = e^{-\int_T^{T'} r_s ds}.$$

The relation (45) is justified by considering spot prices. The random variable $\kappa A'_T$ is nothing but the spot price at time T of the second-period allowance. Because futures and spot price agree at maturity, A_T must be the spot price of the first period allowance at T. In the case of compliance in the first period, the unused allowances can be banked, hence we have the equality in (45).

In the case of non-compliance at the end of the first period, the withdrawal regulation implies that

$$A_T 1_N = \kappa A'_T 1_N + \pi 1_N. \tag{46}$$

Namely, the non-compliance in one pollutant unit at time T costs a penalty π in addition to one allowance from the next period which must be withdrawn at the spot price $\kappa A'_T$.

Combining the results (45) and (46) we find out that the difference is

$$A_t - \kappa A'_t = \mathbb{E}^{\mathbb{Q}}(A_T - \kappa A'_T \,|\, \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(\pi 1_N \,|\, \mathcal{F}_t) \qquad t \in [0, T]$$

and must be modeled as $\{0, \pi\}$-valued martingale. We suggest to use the same methodology as in one period model

$$A_t - \kappa A'_t = \pi \Phi(X_t^1) \qquad t \in [0, T], \tag{47}$$

where the Gaussian process $(X_t^1)_{t\in[0,T]}$ is introduced as previously in (12), with $(\sigma_s)_{s\in[0,T]}$ in parameterized form (28) and driven by a process $(W_t^1, \mathcal{F}_t)_{t\in[0,T']}$of Brownian motion.

To model the second-period allowance futures price, a continuation of the cap-and-trade system must be specified. If there is no agreement on long-term regulatory framework (as it is the case for the most of the existing emission markets), the process $(A'_t)_{t\in[0,T]}$ should be specified exogenously. The simplest choice would be a Geometric Brownian motion with constant volatility. Another idea to handle the uncertain continuation is to suppose that the cap and trade system will be terminated after the second period. In this case,

$$A'_t = \mathbb{E}^{\mathbb{Q}}(\pi 1_{N'} \,|\, \mathcal{F}_t) \qquad t \in [0, T']$$

can also be modeled as in the one-period model

$$A'_t = \pi \Phi(X_t^2) \qquad t \in [0, T']. \tag{48}$$

Again, $(X_t^2)_{t\in[0,T]}$ is introduced as in (12), with a process $\{\sigma_s^2\}_{s\in[0,T]}$ chosen in parameterized form (28) and driven by another Brownian motion $(W_t^2, \mathcal{F}_t)_{t\in[0,T']}$.

5.2 Option Pricing

As an application of our two-period model, we consider pricing of European Calls. Consider European Call option with strike price $K \geq 0$ and maturity $\tau \in [0, T]$ written on futures price of allowance from the first period. This contract yields a payoff

$$C_\tau = (A_\tau - K)^+ \qquad \text{at time } \tau \in [0, T].$$

Under the assumptions of the previous section, we start with the computation of the price C_0

$$C_0 = e^{-\int_0^\tau r_s ds} \mathbb{E}^{\mathbb{Q}}((A_\tau - K)^+)$$

of this option at time $t = 0$. Using the decomposition

$$(A_\tau - K)^+ = (A_\tau - \kappa A'_\tau + \kappa A'_\tau - K)^+,$$

we utilize our modeling of $\{0, \pi\}$-valued martingales (47) and (48) to express the terminal payoff as

$$(A_\tau - K)^+ = (\pi\Phi(X_\tau^1) + \kappa\pi\Phi(X_\tau^2) - K)^+$$

with expectation

$$\begin{aligned} C_0 &= e^{-\int_0^\tau r_s ds} \mathbb{E}^{\mathbb{Q}}((A_\tau - K)^+) \\ &= e^{-\int_0^\tau r_s ds} \mathbb{E}^{\mathbb{Q}}((\pi\Phi(X_\tau^1) + \kappa\pi\Phi(X_\tau^2) - K)^+) \\ &= e^{-\int_0^\tau r_s ds} \int_{\mathbb{R}^2} (\pi\Phi(x_1) + \kappa\pi\Phi(x_2) - K)^+ N(\mu_\tau, \nu_\tau)(dx_1, dx_2) \end{aligned} \tag{49}$$

where $N(\mu_\tau, \nu_\tau) = F_{X_\tau^1, X_\tau^2}$ stands for joint normal distribution of X_τ^1 and X_τ^2.

Let us derive the mean μ_τ and the covariance matrix ν_τ under the standing assumption $\alpha_1 = \alpha_2 = 1$, for $\beta_1 > 0$, $\beta_2 > 0$. We have

$$\begin{aligned} X_\tau^1 &= \Phi^{-1}(\frac{A_0 - \kappa A'_0}{\pi}) \sqrt{(\frac{T}{T - \tau})^{\beta_1}} + \beta_1^{\frac{1}{2}} \frac{\int_0^\tau (T - u)^{\frac{\beta_1 - 1}{2}} W_u^1 du}{(T - \tau)^{\frac{\beta_1}{2}}} \\ X_\tau^2 &= \Phi^{-1}(\frac{\kappa A'_0}{\pi}) \sqrt{(\frac{T'}{T' - \tau})^{\beta_2}} + \beta_2^{\frac{1}{2}} \frac{\int_0^\tau (T' - u)^{\frac{\beta_2 - 1}{2}} W_u^2 du}{(T' - \tau)^{\frac{\beta_2}{2}}}. \end{aligned}$$

Denoting by ρ the correlation of the two Brownian motions $(W_t^1)_{t \in [0, T']}$ and $(W_t^2)_{t \in [0, T']}$

$$[W^1, W^2]dt = \rho dt, \qquad \rho \in [-1, 1],$$

we can apply the same argumentation to obtain the means

$$\begin{aligned} \mu_\tau^1 &= E(X_\tau^1) = \Phi^{-1}(\frac{A_0 - \kappa A'_0}{\pi}) \sqrt{(\frac{T}{T - \tau})^{\beta_1}}, \\ \mu_\tau^2 &= E(X_\tau^2) = \Phi^{-1}(\frac{\kappa A'_0}{\pi}) \sqrt{(\frac{T'}{T' - \tau})^{\beta_2}}, \end{aligned}$$

the variances

$$\nu_\tau^{1,1} = \mathrm{Var}(X_\tau^1) = \left(\frac{T}{T-\tau}\right)^{\beta_1} - 1,$$
$$\nu_\tau^{2,2} = \mathrm{Var}(X_\tau^2) = \left(\frac{T'}{T'-\tau}\right)^{\beta_2} - 1,$$

and the covariance as

$$\nu_\tau^{1,2} = \nu_\tau^{2,1} = \mathrm{Cov}(X_\tau^1, X_\tau^2) = \beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}} \frac{\int_0^\tau (T-u)^{\frac{\beta_1-1}{2}}(T'-u)^{\frac{\beta_2-1}{2}}\rho du}{(T-\tau)^{\frac{\beta_1}{2}}(T'-\tau)^{\frac{\beta_2}{2}}}.$$

At times $t \in [0, \tau]$ prior to maturity, the price C_t of the call is obtained similarly:

Proposition 3. *In a two-compliance periods model as above, with parameters $\beta_1, \beta_2 > 0$ and $\rho \in (-1, 1)$, the price of the European Call with strike price $K \geq 0$ and maturity $\tau \in [0, T]$ written on first-period allowance futures price is given at time $t \in [0, \tau]$ by*

$$C_t = e^{-\int_t^\tau r_s ds} \int_{\mathbb{R}^2} (\pi\Phi(x_1) + \kappa\pi\Phi(x_2) - K)^+ N(\mu_{t,\tau}, \nu_{t,\tau})(dx_1, dx_2) \tag{50}$$

with mean $\mu_{t,\tau}$

$$\mu_{t,\tau}^1 = \Phi^{-1}\left(\frac{A_t - \kappa A'_t}{\pi}\right)\sqrt{\left(\frac{T-t}{T-\tau}\right)^{\beta_1}} \tag{51}$$
$$\mu_{t,\tau}^2 = \Phi^{-1}\left(\frac{\kappa A'_t}{\pi}\right)\sqrt{\left(\frac{T'-t}{T'-\tau}\right)^{\beta_2}} \tag{52}$$

and covariance matrix $\nu_{t,\tau}$

$$\nu_{t,\tau}^{1,1} = \mathrm{Var}(X_\tau^1) = \left(\frac{T-t}{T-\tau}\right)^{\beta_1} - 1 \tag{53}$$
$$\nu_{t,\tau}^{2,2} = \mathrm{Var}(X_\tau^2) = \left(\frac{T'-t}{T'-\tau}\right)^{\beta_2} - 1 \tag{54}$$
$$\nu_{t,\tau}^{1,2} = \nu_{t,\tau}^{2,1} = \beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}} \frac{\int_t^\tau (T-u)^{\frac{\beta_1-1}{2}}(T'-u)^{\frac{\beta_2-1}{2}}\rho du}{(T-\tau)^{\frac{\beta_1}{2}}(T'-\tau)^{\frac{\beta_2}{2}}}. \tag{55}$$

If we take a closer look at the computations involved in the valuation of the call price

$$C_t = C_t(\tau, T, T', A_0, A'_0, K, r, \beta_1, \beta_2, \rho)$$

given by the formulas (50) – (55), we see that because of

$$C_t(\tau, T, T', A_0, A'_0, K, r, \beta_1, \beta_2, \rho) = C_0(\tau - t, T - t, T' - t, A_0, A'_0, K, r, \beta_1, \beta_2, \rho)$$

for all $t \in [0, \tau]$, it suffices to consider the case $t = 0$. The numerical evaluation of two dimensional integral is easily performed by using a decomposition of the two-dimensional normal distribution. To ease the notation, let us skip t, τ to write $\mu^i = \mu^i_{t,\tau}$, $\nu^{i,j} = \nu^{i,j}_{t,\tau}$ for $i, j = 1, 2$. It holds

$$N(\mu, \nu)(dx_1, dx_2) = N(\mu^{1,c}(x_2), \nu^{1,1,c})(dx_1)N(\mu^2, \nu^{2,2})(dx_2) \tag{56}$$

where the conditional mean and the conditional variance are given by

$$\begin{aligned} \mu^{1,c}(x_2) &= \mu^1 + \frac{\nu^{2,1}}{\nu^{2,2}}(x_2 - \mu^2) \\ \nu^{1,1,c} &= \nu^{1,1} - \frac{(\nu^{2,1})^2}{\nu^{2,2}} \end{aligned}$$

With factorization (56), the inner integral is calculated explicitly in the following cases

$$\int_{\mathbb{R}} (\pi\Phi(x_1) + \kappa\pi\Phi(x_2) - K)^+ N(\mu^{1,c}(x_2), \nu^{1,1,c})(dx_1)$$
$$= \begin{cases} 0 & \text{if} \quad K - \pi\kappa\Phi(x_2) \geq \pi \\ \pi\Phi(\frac{\mu^{1,c}(x_2)}{\sqrt{1+\nu^{1,1,c}}}) + \pi\kappa\Phi(x_2) - K & \text{if} \quad K - \pi\kappa\Phi(x_2) \leq 0 \end{cases}.$$

That is, the numerical valuation is required only in the case $0 < K - \pi\kappa\Phi(x_2) < \pi$ where

$$\int_{\Phi^{-1}(K/\pi - \kappa\Phi(x_2))}^{\infty} (\pi\Phi(x_1) + \kappa\pi\Phi(x_2) - K)N(\mu^{1,c}(x_2), \nu^{1,1,c})(dx_1)$$

needs to be calculated.

Having obtained the inner integral, the numerical evaluation of the outer integral is straightforward. Since the density of the normal distribution decays sufficiently fast, we do not expect neither numerical difficulties nor long computation times. In fact, we did not encounter any problem implementing this formula.

For the sake of completeness, we illustrate the dependence of the call price on β_1 and maturity of the call. To make the results comparable with the one-period example given above, we chose the following parameters: four years to the first-period compliance date $T = 4$, eight years to the second-period compliance date $T_2 = 8$, initial first-period allowance futures price is $A_0 = 25$, initial second-period allowance futures price is $A'_0 = 15$, strike price of the European call is $K = 25$, interest rate $r = 0.05$, and $\beta_2 = 0.2$. Figure 5.2 depicts the dependence of the call price on the value of β_1 for the first period and of the call maturity τ.

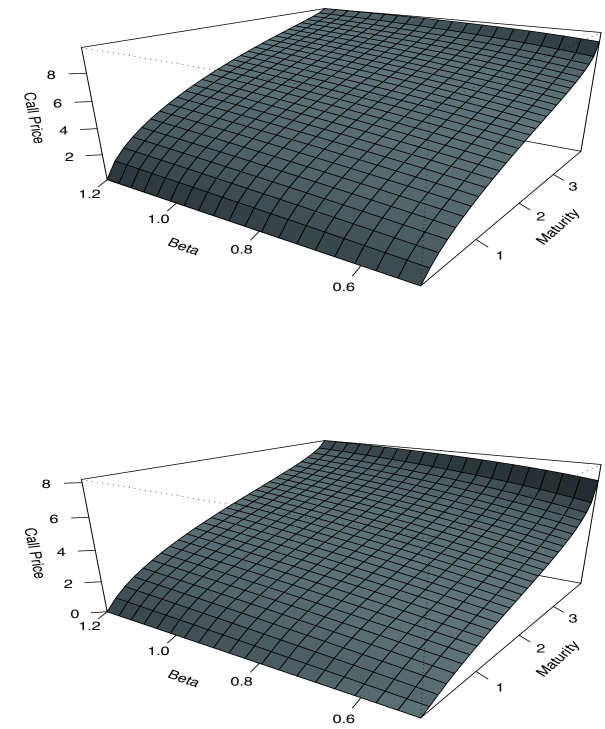

Figure 6: Surface plots of the initial call price $(\tau, \beta_1) \mapsto C_0(\tau, \beta_1)$ as function of maturity τ and β_1 for correlation $\rho = 0.8$ (top) and $\rho = -0.8$ (bottom). The values of the other parameters are given in the text.

6 Conclusion

Mandatory emission markets are being established throughout the world. In the most mature market, the European Emission Trading Scheme, beyond physical allowances, a large volume of allowance futures is traded. Furthermore, European options written on these futures are introduced and traded although no theoretical foundation for their pricing is available yet.

The goal of this work is to fill this gap. In our analysis, we gradually move from one-period market model to a more realistic situation of two-period markets (covering the present EU ETS regulations) and show that martingales finishing at two-valued random variables can be considered as basic building blocks which form the risk-neutral futures price dynamics. We suggest a model for two-valued martingales, flexible in terms of time- and space changing volatility and capable to match the observed historical or implied volatility of the underlying future. From hedging perspective, this issue could be one of the most desirable model properties. Other practical aspects like ease of calibration and simple option valuation schemes are also fulfilled in our approach. We show how parameters can be estimated from historical price observation and suggest efficient option valuation schemes. Although option price formulas are not available in a closed form, a simple and fast numerical integration can be applied.

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