Noname manuscript No.
(will be inserted by the editor)

Tangent Lévy Market Models

René Carmona · **Sergey Nadtochiy**

the date of receipt and acceptance should be inserted later

Abstract In this paper, we introduce a new class of models for the time evolution of the prices of call options of all strikes and maturities. We capture the information contained in the option prices in the density of some time-inhomogeneous Lévy measure (an alternative to the implied volatility surface), and we set this static code-book in motion by means of stochastic dynamics of Itôs type in a function space, creating what we call a *tangent Lévy model*. We then provide the consistency conditions, namely, we show that the call prices produced by a given dynamic code-book (*dynamic Lévy density*) coincide with the conditional expectations of the respective payoffs if and only if certain restrictions on the dynamics of the code-book are satisfied (including a drift condition à la HJM). We then provide an existence result, which allows us to construct a large class of tangent Lévy models, and describe a specific example for the sake of illustration.

Keywords Implied volatilty surface - Tangent models - Lévy Processes - Market models - Arbitrage-free term structure dynamics - HeathJarrowMorton theory

Mathematics Subject Classification (2000) 91B24

1 Introduction

The classical approach to modeling prices of financial instruments is to identify a certain (small) family of ”underlying” processes, whose dynamics are described explicitly, and compute the prices of the financial derivatives written on these underliers by taking expectations under the risk-neutral measure or maximizing an expected utility. Such is the famous Black-Scholes model, where the underlying stock price is assumed to be given by geometric

Partially supported by NSF Grant #180-6024

R. Carmona
Bendheim Center for Finance, ORFE, Princeton University, Princeton, NJ 08544, E-mail: rcarmona@princeton.edu

S. Nadtochiy
Oxford-Man Institute of Quantitative Finance University of Oxford, E-mail: sergey.nadtochiy@oxford-man.ox.ac.uk

Brownian motion. On the contrary, the present paper is concerned with the construction of so-called *market models* which describe the simultaneous dynamics of all the liquidly traded derivative instruments. The new family of models proposed in this paper can be viewed as an extension of the results of [3] which should be consulted for a more detailed discussion of the history of the "market model" approach.

As it was done in [3], we limit ourselves to a single underlying index or stock on which all the derivatives under consideration are written. We also assume that the discount factor is one, or equivalently that the short interest rate is zero, and that the underlying security does not pay dividends. These assumptions greatly simplify the notation without affecting the generality of our derivations as long as the interest and dividend rates are deterministic.

We assume that in our idealized market European call options of all strikes and maturities are traded, that their prices are observable, and that they can be bought and sold at these prices in any quantity. We denote by $C_t(T,K)$ the market price at time t of a European call option of strike K and maturity $T > t$. We assume that today, i.e. on day $t = 0$, all the prices $C_0(T,K)$ are observable. According to the philosophy of market models adopted in this paper, at any given time t, instead of modeling only the price S_t of the underlying asset, we use the set of call prices $\{C_t(T,K)\}_{T \geq t, K \geq 0}$ as our fundamental market data. This is partly justified by the well documented fact that many observed option price movements cannot be attributed to changes in S_t, and partly by the fact that many exotic (path dependent) options are hedged (replicated) with portfolios of plain (vanilla) call options. In this context, it becomes important to have a model that is consistent with the market prices of vanilla options. However, it is well known that the Black-Scholes model does not reproduce prices of call options with different strikes and maturities faithfully. This phenomenon is sometimes referred to as the "implied smile" effect. *Stochastic volatility models* containing more parameters, can be calibrated to match at least approximately, a finite set of observed option prices and solve the "implied smile" problem in a rather satisfactory manner. However, the calibration has to be done at the beginning of each trading period, implying computational complexity and a lack of time-consistency in the model: as time passes by, not only does the value of the underlying index change, but the values of the calibrated parameters also change, even though they are assumed to be constant by the model. On the contrary, models from the family of market models introduced in this paper are automatically consistent with observed option prices, since these prices become a part of the initial condition for the dynamics of the model.

Early attempts to construct market models for vanilla options can be found in [16], [9] and [10]. This idea was then developed more thoroughly in the works of Schönbucher [32], Schweizer and Wissel [34] and Jacod and Protter [21], but the recent works of Schweizer and Wissel [33] and Carmona and Nadtochiy [3], [2] are more in the spirit of the market model approach that we advocate here.

The first hurdle on the way to creating a stochastic dynamic model for the call price surface (price is considered as a function of strike and maturity) is to describe its state space. Clearly, not every nonnegative function of two variables can be a surface of call prices – there are conditions it has to satisfy: for example, prices should converge to the payoff as time to maturity goes to zero. In addition, there are so-called "static no-arbitrage" conditions: a call price is a nondecreasing function of maturity and a nonincreasing and convex function of strike (see [26], [13], [1] and [15] for more on this). Notice that these (necessary) conditions can be violated by a "small" (in the sense of corresponding norm) perturbation of the surface, which implies that the set of admissible call price surfaces cannot be defined as an open subset of a linear space. In a sense, this set forms a manifold in the infinite dimensional space of functions of two variables. However, since we would like to model the time evolution of call prices through a system of stochastic differential equations (SDE's), it becomes necessary to have some kind of differential calculus on this manifold. Differentiation on a manifold is usually done via mapping it into a linear space, where the differential calculus is well developed. Therefore, in order to describe the state space, we need to find the right parametrization for the surface of option prices, or in other words, the right *code-book*.

In [3] we proposed the *local volatility* as a code-book for option prices. Defining the local volatility through Dupire's formula (see [18]), one can obtain a correspondence between the local volatility and option prices. This correspondence results in a parametrization of a class of admissible call price surfaces, and one important feature of this parametrization is that the new "variable", i.e. the local volatility, has only to be non-negative and to satisfy some mild smoothness conditions in order to produce an admissible call price surface. These properties define open sets in appropriate linear spaces on which the *dynamic local volatility* can then be constructed.

Notice, however, that not every call price surface can be represented via a local volatility surface: for example, it is easy to see that, if the underlying is given by a *pure jump martingale*, the corresponding local volatility surface resulting from the Dupire's formula will explode at short maturities (as $T \searrow t$), and such a surface cannot be used to reproduce the call prices in this case. Then two questions arise naturally: "*what is the set of call price surfaces which can be reproduced by local volatility models*?" and "*what are the other possible code-books which can be used when local volatility can't*?" The first question has been answered by Gyongy [19], who showed that, in the case when underlying follows a regular enough Itô process, the local volatility can be used to reproduce the call prices. In accordance with this result, the underlying in [3] was assumed to be a continuous Itô process satisfying some regularity conditions. Addressing the second question, one would first ask: besides relaxing the technical conditions, what is a possible extension of these assumptions on the underlying index? Staying within the class of semimartingales, we can only introduce jumps.

In this paper we assume that the risk-neutral dynamics of the option underlier are given by a *pure jump martingale* and we argue that the right substitute for the local volatility, as a code-book for option prices, can be based on a specific Lévy measure. We assume that at any given time, the surface of call prices can be recovered by the use of an *additive* (inhomogeneous Lévy) process. Since the distribution of such a process is completely characterized by its Lévy measure, assuming that this measure is absolutely continuous, we end up capturing the information contained in the call prices in the density of a (time-inhomogeneous) Lévy measure. This point of view is static since it leads to the analysis of the option prices at a fixed point in time. But like in [3] and [2], our goal is to construct market models by putting in motion the static code-book chosen to describe the option prices. So, at each fixed time, our pure jump martingale model for the underlying asset will have to produce the same option prices as the static model given by the additive process with Lévy density being the current value of the code-book. Therefore, just like in the case of dynamic local volatility models treated in [3] and [2], with each call price surface we associate a process from a parameterized family of "simple" (exponential additive, in the present case) processes which reproduce the observed option prices, and then model the time evolution of the parameter value (density of the Lévy measure), obtaining a market model. So, at each fixed time, our pure jump martingale model for the underlying asset admits a form of tangent Lévy process, in the sense that locally (at the current point in time) both processes produce the same option prices. This is the reason for our terminology of *tangent Lévy model*. This class of pure jump martingales should not be confused with the class of processes admitting an additive tangent process in the sense introduced by Jacod in [20] and further studied in [22], in his attempt to generalize the notion of semi-martingale.

The idea of using processes with jumps to model the prices of financial assets has a long history and dates back to Merton [29] who first introduced jumps in the stock price dynamics in 1976. The extension provided by Kou's double exponential jump diffusion model (see [24]) produces closed form expressions not only for the prices of European options but also for some exotic derivatives. A number of papers by Carr, Geman, Madan and Yor were devoted to the use of Lévy processes for pricing derivatives. Probably, the most popular one is the CGMY model (see [5]), which is an extension of the Variance Gamma model introduced in [28]. In this model, the logarithm of the underlying index is assumed to follow a pure jump Lévy process whose Lévy density, separately for positive and negative jump size x, is given by a scaled ratio of decaying exponential over a power of $|x|$. The pure jump exponential Lévy models allow for implied smile and heavy tails in the log-return distribution, and they, clearly, fit the option prices better than the Black-Scholes model. It is, however, worth mentioning that the above models are of the classical type, in the sense that their main idea is to describe precisely the risk-neutral dynamics of the underlying process and compute the prices of derivatives by taking expectations. The framework developed in this paper is dictated by the market model approach, and, therefore, the resulting models are fundamentally different from the ones described above: in particular, they allow for much more general dynamics of the underlying than the exponential Lévy processes.

In 2004 Carr, Geman, Madan and Yor [6] proposed a way to reproduce option prices of all strikes and maturities by a time changed Lévy process, introducing the *local Lévy* models. These authors constructed the *local speed function* as an analogue of local volatility for pure jump models. Their paper served as an inspiration for the present work, even though we do not use the local speed function. Instead, we propose a different, more convenient, code-book in lieu of local volatility.

We close this introduction with a quick summary of the contents of the paper. Section 2 introduces the code-book designed to capture the information contained in the surface of call options. In doing so, we precise the type of non-homogeneous Lévy processes (also called additive processes) which we use to reproduce call prices at any given time. The class of pure jump martingales providing the risk neutral dynamics of the underlying asset, together with the definition of tangent Lévy models are presented in Section 3. There, we explain how the static code-book, given by the time-inhomogeneous Lévy density, is set in motion by means of a stochastic dynamics of Itô's type in a function space. Section 4 is devoted to the derivation of the consistency conditions: the necessary and sufficient conditions for a given dynamic Lévy density and an underlying process to form a tangent Lévy model. These conditions are formulated explicitly in terms of the semimartingale characteristics of the processes (including a drift restriction à la HJM). Finally, we prove existence of a large class of tangent Lévy models in Section 5. We construct explicit examples and briefly discuss their implementation in Section 6. Two short appendices are devoted to the technical proofs of results needed throughout the paper.

2 Preliminaries

In this section we summarize the results on *additive processes*, which we subsequently use to construct new code-books for the call price surfaces.

2.1 Background on Additive Processes

Additive processes are Lévy processes without time homogeneity, so most of their properties can be derived from the results known for Lévy processes. Let us denote by $\left(\tilde{S}_T\right)_{T\geq 0}$ the *exponential additive* pure jump martingale, given by the solution of the following stochastic integral equation:

$$\tilde{S}_T = \tilde{S}_0 + \int_0^T \int_{\mathbb{R}} \tilde{S}_{u-}(e^x - 1)(\tilde{N}(dx, du) - \tilde{\eta}(dx, du)), \tag{1}$$

where $\tilde{N}(dx, du)$ is a *Poisson random measure* (associated with the jumps of the logarithm of the process) which has the following deterministic compensator

$$\tilde{\eta}(dx, du) = \kappa(u, x)dxdu. \tag{2}$$

Definition (1) looks indeed like an equation for $\tilde{S}$, but, in fact, a simple application of Itô's rule shows that the solution is given by $\tilde{S}_T = \exp \tilde{X}_T$, with

$$\tilde{X}_T = \log \tilde{S}_0 - \int_0^T \int_{\mathbb{R}} \left(e^x - x - 1\right) \tilde{\eta}(dx, du) + \int_0^T \int_{\mathbb{R}} x(\tilde{N}(dx, du) - \tilde{\eta}(dx, du)) \tag{3}$$

being an additive process (which explains the terminology "exponential additive"). In order for the expressions above and the derivations that follow to make sense, we need to assume that the Lévy density κ satisfies

$$\int_0^T \int_{\mathbb{R}} (|x| \wedge 1)|x|(1 + e^x)\kappa(u, x)dxdu < \infty, \qquad t > 0. \tag{4}$$

Let us assume for a moment that $0 \leq t < T$ are fixed. Then, for each bounded Borel subset B of $\mathbb{R}$, the random variable $\tilde{N}\,(B \times [t, T])$ has the same distribution as $\hat{N}\,(B \times [t, T])$, where $\hat{N}$ is a time-homogeneous Poisson random measure given by its Lévy measure

$$\hat{\eta}(dx) = \frac{1}{T - t} \left(\int_t^T \kappa(u, x)du \right) dx. \tag{5}$$

Therefore, the conditional distribution of $\tilde{X}_T$ given $\tilde{X}_t = x$ is the same as the distribution at time $T - t$ of a Lévy process which starts from x at time 0, and has Lévy measure $\hat{\eta}$. If, for $t = 0$ and $x = \log \tilde{S}_0$, we denote such a process by $\hat{X}$, we can apply the classical theory developed for Lévy processes (see for example Theorem 25.3 and 25.17 in [31]) to conclude that

$$\mathbb{E}\tilde{S}_T = \mathbb{E} \exp \hat{X}_T = \exp \hat{X}_0 = \tilde{S}_0, \tag{6}$$

which is true for any $T > 0$. Notice also that, by definition, $\tilde{S}$ is the stochastic (Doléans-Dade) exponential of the process $\tilde{Y}$ defined by

$$\tilde{Y}_T = \log \tilde{S}_0 + \int_0^T \int_{\mathbb{R}} (e^x - 1)(\tilde{N}(dx, du) - \tilde{\eta}(dx, du)).$$

The above observations yield that $\tilde{S}$ is a positive local martingale, which, together with (6), implies that $\tilde{S}$ is a true martingale by a standard argument. This fact is also mentioned on p. 460 of [11].

2.2 Option Prices in Exponential Additive Models

We now consider a financial market consisting of a single underlying instrument, assume that the interest rates are zero and pricing is done via expectations under a risk-neutral measure. We denote the level of the underlying index at time t by S_t. For the rest of this section, time t is fixed and S_t should be viewed as a fixed positive real number (we will give prescriptions for its stochastic dynamics in the subsequent sections). Then, in a hypothetical model, in which from time t on the underlying risk-neutral dynamics are given by $\tilde{S}$, defined in (1), and the market filtration is generated by $\tilde{S}$, the time t price of a call option with strike $K = e^x$ and maturity T is given by

$$C_t^{S_t,\kappa}(T, x) = \mathbb{E}\left[\left(\tilde{S}_T - e^x\right)^+ \middle| \tilde{S}_t = S_t \right]. \tag{7}$$

It is clear that the above call prices are uniquely determined by the conditional distribution of $\left(\tilde{S}_u\right)_{u\in[t,T]}$, given $\tilde{S}_t = S_t$, which in turn, depends only upon S_t and κ. This justifies the notation $C^{S_t,\kappa}$.

It is important to keep in mind the fact that the model given by (1) **is not** the actual model for the underlying asset which we propose and study in this paper!

The rest of this section is devoted to the derivation of analytic expressions for the call prices (7) in terms of the Lévy density κ of the process $\left(\tilde{S}_u\right)_{t\leq u\leq T}$. Notice that, although the derivation of equations (10) and (12) below is heuristic, a rigorous proof of the resulting formula (13) is given by (14) and references listed in the subsequent paragraph.

Repeating essentially the derivations from [6] or [12], we obtain the following Partial Integro-Differential Equation (PIDE) for the call prices (see, for example, equation (13) in [6])

$$\begin{cases} \partial_T C_t^{S_t,\kappa}(T,x) = \int_{\mathbb{R}} \psi(\kappa(T,\cdot\,); x-y) D_y C_t^{S_t,\kappa}(T,y)dy \\ \qquad\qquad C_t^{S_t,\kappa}(t,x) = (S_t - e^x)^+, \end{cases} \tag{8}$$

where D_x denotes the second order partial differential operator $D_x = \partial^2_{x^2} - \partial_x$ and

$$\psi(f;x) := \begin{cases} \int_{-\infty}^x (e^x - e^z) f(z)dz \;\; x < 0 \\ \int_x^\infty (e^z - e^x) f(z)dz \;\; x > 0, \end{cases} \tag{9}$$

is the double exponential tail function introduced in [6]. We will sometimes write $\psi(f(T);x)$ instead of $\psi(f(T,\cdot\,);x)$ when the function f has two arguments.

The initial value problem (8) involves constant coefficient partial differential operators and convolutions, so it is natural to use Fourier transform. Unfortunately, the function giving the initial condition in problem (8) is not integrable on $\mathbb{R}$, hence its Fourier transform is not well defined as a function in the classical sense. In order to resolve this problem, we rewrite (8), differentiating both sides with respect to the "log-strike"variable x (see [7] for the alternative approach). Using the notation $\Delta_t(T,x) = -\partial_x C_t^{S_t,\kappa}(T,x)$, we have

$$\begin{cases} \partial_T \Delta_t(T,x) = \int_{\mathbb{R}} \psi(\kappa(T); x-y) D_y \Delta_t(T,y) dy \\ \Delta_t(t,x) = e^x \mathbf{1}_{(-\infty, \log S_t]}(x), \end{cases} \tag{10}$$

We chose to use the Greek letter delta as it is, at least in finance, the standard notation for the derivative of the price of an option with respect to the underlying value or the strike. Because of the presence of the two arguments T and x, we believe that this choice will not create confusion with the use of Δ for the Laplacian or second derivative. The initial condition of the above problem being in $\mathbb{L}^1(\mathbb{R})$, we can solve (10) in the Fourier domain. As a general rule, we shall use a superscript "hat" for the direct Fourier transform, and a "check" for the inverse Fourier transform. In particular

$$\hat{\psi}(f;\xi) := \int_{\mathbb{R}} e^{-2\pi i x \xi} \psi(f;x) dx. \tag{11}$$

Problem (10) becomes

$$\begin{cases} \partial_T \hat{\Delta}_t(T,\xi) = \hat{\psi}(\kappa(T);\xi) \hat{\Delta}_t(T,\xi) \left(-4\pi^2 \xi^2 - 2\pi i \xi\right) \\ \hat{\Delta}_t(t,\xi) = \frac{e^{\log S_t (1-2\pi i \xi)}}{1-2\pi i \xi} \end{cases} \tag{12}$$

As a side remark we notice that the first equation above gives a mapping from the call prices (as given by $\hat{\Delta}$) to κ (as given by $\hat{\psi}$). We continue deriving analytic expressions for call prices in terms of κ. Solving (12), we obtain

$$\hat{\Delta}_t(T,\xi) = \frac{e^{\log S_t (1-2\pi i \xi)}}{1-2\pi i \xi} \exp\left(-2\pi(2\pi\xi^2 + i\xi) \int_{T\wedge t}^{T} \hat{\psi}(\kappa(u);\xi) du\right), \tag{13}$$

where we employ the notation

$$a \wedge b := \min(a,b), \quad a \vee b := \max(a,b),$$

which will be used throughout the paper. Notice that in this section, the maturity T is never smaller than the current calendar time t, and, therefore, $T \wedge t = t$. However, since (13) will be referenced in the subsequent sections, where the domain of the T-variable does not depend upon t, we need (13) to be well defined for $t > T$. Notice now that, as shown in Appendix A, the following equality holds

$$\exp\left(-2\pi(2\pi\xi^2 + i\xi) \int_{T\wedge t}^{T} \hat{\psi}(\kappa(u);\xi) du\right) = \mathbb{E}\left(e^{(1-2\pi i \xi)\log \tilde{S}_T} \middle| \log \tilde{S}_t = 0\right), \tag{14}$$

As mentioned earlier, the distribution of $\log \tilde{S}_T$, conditioned by $\log \tilde{S}_t = \log S_t$, is the same as the marginal distribution at time $T-t$ of a Lévy process that starts from $\log S_t$ at time 0 and has Lévy measure (5). Exponential Lévy models in finance have been studied rather thoroughly, and several methods for the computation of option prices have been proposed. In the present situation, equality (14) establishes an equivalence between (13) and the well known formula for the Fourier transform of call prices in the exponential Lévy models, derived in [7] and also stated in [11] (see, for example, equation (14) in [7] or equation (11.19) in [11]). This simple observation provides a rigorous proof of (13).

It also follows from the representation formula (14) that, for all $\xi \in \mathbb{R}$,

$$\left| \exp\left(-2\pi(2\pi\xi^2 + i\xi) \int_{T\wedge t}^{T} \hat{\psi}(\kappa(u);\xi)du \right) \right| \leq \mathbb{E}\left(\tilde{S}_T \middle| \tilde{S}_t = 1 \right) = 1, \tag{15}$$

which implies that $\hat{\Delta}_t(T,\,\cdot\,) \in \mathbb{L}^2(\mathbb{R})$. The Fourier transform and its inverse are well defined and unitary on this space. In particular, inverting the Fourier transform and integrating, one can obtain the following expression for $C_t^{S_t,\kappa}(T,x)$:

$$C_t^{S_t,\kappa}(T,x) = \tag{16}$$
$$S_t \lim_{\lambda\to+\infty} \int_{\mathbb{R}} \frac{e^{2\pi i\xi\lambda} - e^{2\pi i\xi(x-\log S_t)}}{2\pi i\xi(1-2\pi i\xi)} \exp\left(-2\pi(2\pi\xi^2 + i\xi) \int_{t\wedge T}^{T} \hat{\psi}(\kappa(u);\xi)du \right) d\xi.$$

The purpose of formula (16) is not to provide the most efficient method for the computation of call prices in the exponential *Lévy* and *additive* models. The interested reader is referred to [7], [11] and the references therein for more on such methods. In fact, for the derivations that follow, formula (13) is the most convenient analytic representation of the call prices in exponential additive models, and it will be used in the subsequent sections. We chose to provide equation (16) only for the sake of completeness and in order to highlight the difficulties associated with it (see the paragraph following the proof of Proposition 6).

3 Tangent Lévy Models

In this section we introduce the family of models studied in this paper. From now on, we fix $\bar{T} > 0$ and we consider only $t \in [0,\bar{T}]$. We work with a stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{Q})$, the filtration $\mathbb{F}$ satisfying the *usual hypotheses* (see definitions I.1.2 and I.1.3 in [23]), and on which all the random processes introduced below are defined.

Our financial market consists of a single underlying asset whose price is given by an adapted semimartingale $(S_t)_{t\in[0,\bar{T}]}$, and we assume that European call options with all possible strikes $K = e^x$ and maturities $T \in (t,\bar{T}]$ are available for trade at time t at the price $C_t(T,x)$ given by the conditional expectation under $\mathbb{Q}$ of the payoff at maturity T.

As explained in Section 1, we are interested in constructing a class of models in which call prices have explicit and flexible dynamics. Namely, we assume that, at each point in time t, there exists a nonnegative function $\kappa_t(\,\cdot\,,\,\cdot\,)$, such that the call prices are given by $C_t^{S_t,\kappa_t}(T,x)$ defined in (7). We emphasize that the surface κ_t characterizing the call prices, is different at each instant t, explaining why we now add the time as a subscript. With the above convention, we can model explicitly the joint dynamics of κ_t and S_t through a system of stochastic differential equations, which in turn, produce the dynamics of the call prices. Clearly, one needs to make sure that the dynamics of S_t and κ_t are such that the two "definitions" of the call prices are consistent with each other, namely, make sure that the call prices produced by κ are indeed the conditional expectations of the corresponding payoffs. This results in the *consistency conditions*, which take the form of restrictions on the characteristics of S and κ and are formulated explicitly in Theorem 12 in Section 4. The rest of this section is mostly concerned with defining a priori dynamics of κ_t and S_t.

3.1 Function Spaces

First, we choose a state space for the stochastic process $\kappa = (\kappa_t)_{t\in[0,\bar{T}]}$. Recall that all it has to satisfy in order to produce feasible call prices, besides nonnegativity, is (4). We introduce the Banach space $\mathcal{B}^0$ of equivalence classes of Borel measurable functions $f : \mathbb{R} \hookrightarrow \mathbb{R}$ satisfying

$$\|f\|_{\mathcal{B}^0} := \int_{\mathbb{R}} (|x| \wedge 1)\, |x|(1+e^x)|f(x)|dx < \infty.$$

Next, we define the Banach space $\mathcal{B}$ of *absolutely continuous* functions $f : [0,\bar{T}] \hookrightarrow \mathcal{B}^0$ satisfying

$$\|f\|_{\mathcal{B}} := \|f(0)\|_{\mathcal{B}^0} + \int_0^{\bar{T}} \left\| \frac{d}{du} f(u) \right\|_{\mathcal{B}^0} du < \infty.$$

Recall that a Borel function $f : [0,\bar{T}] \hookrightarrow \mathcal{B}^0$ is said to be absolutely continuous if there exists a measurable function $g : [0,\bar{T}] \hookrightarrow \mathcal{B}^0$, such that for any $t \in [0,\bar{T}]$ we have

$$f(t) := f(0) + \int_0^t g(u)du,$$

where the above integral is understood as the *Bochner integral* (see p. 44 in [17] for a definition) of a $\mathcal{B}^0$-valued function. In such a case, the equivalence class of such functions g is denoted $\frac{d}{dt}f$. In order to check that the definition of $\mathcal{B}$ makes sense, it is enough to notice that the space $\mathbb{L}^1\left(\text{Leb}_{[0,\bar{T}]}, \mathcal{B}^0\right)$ of equivalence classes of integrable $\mathcal{B}^0$-valued functions defined almost everywhere, equipped with its natural norm, is a Banach space (see Section II.2 of [17]). For the sake of convenience we will often say that a function f of two variables, $(t,x) \mapsto f(t,x)$, belongs to $\mathcal{B}$, if the function $\tilde{f}$ defined by $\tilde{f}(t) := f(t,\,\cdot\,)$ for all t, is an element of $\mathcal{B}$.

Clearly, κ_t should be in $\mathcal{B}$. However, in order to apply Itô's formula, we need a *conditional Banach space* (see III.5.3 in [25] for definition). With this in mind, we introduce the Hilbert space $\mathcal{H}^0$ of equivalence classes of functions satisfying

$$\|f\|^2_{\mathcal{H}^0} := \int_{\mathbb{R}} |x|^4(1+e^x)^2|f(x)|^2dx < \infty$$

(the inner product of $\mathcal{H}^0$ being obtained by polarization), and the Hilbert space $\mathcal{H}$ of absolutely continuous functions $f : [0,\bar{T}] \hookrightarrow \mathcal{H}^0$ satisfying

$$\|f\|^2_{\mathcal{H}} := \|f(0)\|^2_{\mathcal{H}^0} + \int_0^{\bar{T}} \left\| \frac{d}{du} f(u) \right\|^2_{\mathcal{H}^0} du < \infty.$$

It is not hard to establish (via iterative use of Cauchy's inequality) that $\mathcal{H}^0 \subset \mathcal{B}^0$, $\mathcal{H} \subset \mathcal{B}$ and $\|\cdot\|_{\mathcal{B}^0} \preceq \|\cdot\|_{\mathcal{H}^0}$, $\|\cdot\|_{\mathcal{B}} \preceq \|\cdot\|_{\mathcal{H}}$, where the notation $\preceq$ means that the natural inclusion of the space on the left into the space on the right is one-to-one with dense range. Clearly, the completion of $\mathcal{H}^0$ in $\|\cdot\|_{\mathcal{B}^0}$ norm is $\mathcal{B}^0$ (since $\mathcal{H}^0$ contains the set of all bounded Borel functions with bounded support, which is dense in $\mathcal{B}^0$), and the completion of $\mathcal{H}$ in $\|\cdot\|_{\mathcal{B}}$ norm is $\mathcal{B}$. Thus, the couple $(\mathcal{H}, \mathcal{B})$ is indeed a conditional Banach space.

3.2 Model Definition

Here we define the components of the model more specifically. In particular, we assume that the risk-neutral evolution of the underlying index is given by $(S_t)_{t\in[0,\bar{T}]}$, which is a cádlág martingale, satisfying, for every $t \in [0,\bar{T}]$, almost surely

$$S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_{u-}(e^x - 1)[M(dx,du) - K_u(x)dxdu], \tag{17}$$

where M is an *integer valued random measure* on $(\mathbb{R}\setminus\{0\}) \times [0,\bar{T}]$ with *compensator* $K_{t,\omega}(x)dxdt$ (see II.1.3, II.1.13 and II.1.8 in [23] for definitions), such that $(K_t)_{t\in[0,\bar{T}]}$ is a predictable integrable stochastic process with values in $\mathcal{B}^0$. Notice that, as follows from the integrability property of the compensator, the measure M satisfies:

$$M\left((\mathbb{R}\setminus(-\varepsilon,\varepsilon)) \times [0,\bar{T}]\right) < \infty,$$

for all $\varepsilon > 0$, and

$$\int_0^{\bar{T}} \int_{\mathbb{R}} (|x| \wedge 1)^2 \, M(dx,du) < \infty$$

almost surely. Formula (17) looks like an equation for S, however, as it was demonstrated in Section 2, a simple application of Itô's rule shows that $S_t = \exp X_t$, where

$$X_t = \log S_0 - \int_0^t \int_{\mathbb{R}} (e^x - x - 1)K_u(x)dxdu + \int_0^t \int_{\mathbb{R}} x[M(dx,du) - K_u(x)dxdu]. \tag{18}$$

Starting from (18), we can work backwards to obtain (17), implying the positivity of S. We now define the dynamics of κ.

Definition 1 A $\mathcal{B}$-valued continuous stochastic process $(\kappa_t)_{t\in[0,\bar{T}]}$ is a **dynamic Lévy density** if, almost surely, for all $t \in [0,\bar{T})$ and $T \in (t,\bar{T}]$

$$\text{ess inf}_{x\in\mathbb{R}} \, \kappa_t(T,x) \geq 0,$$

and the following representation hold almost surely, for all $t \in [0,\bar{T})$

$$\kappa_t = \kappa_0 + \int_0^t \alpha_u du + \sum_{n=1}^m \int_0^t \beta_u^n dB_u^n, \tag{19}$$

where $B = \left(B^1,\ldots,B^m\right)$ is a multidimensional Brownian motion, α is a progressively measurable integrable stochastic process with values in $\mathcal{B}$, and $\beta = \left(\beta^1,\ldots,\beta^m\right)$ is a vector of progressively measurable square integrable stochastic processes taking values in $\mathcal{H}$.

Remark 2 Notice that κ takes values in an infinite dimensional space, therefore, it may seem natural to have an infinite dimensional Brownian motion driving its dynamics. Indeed, it is possible to treat the case of $m = \infty$ by considering the canonical Gaussian measure of some real separable Hilbert space $\bar{\mathcal{H}}$ and its associated *cylindrical Brownian motion* B (see [4] or [25]). The process β in this case would take values in the space of Hilbert-Schmidt operators from $\bar{\mathcal{H}}$ into $\mathcal{H}$, and β_t^n would be the value of β_t on the n-th vector of some orthonormal system in $\bar{\mathcal{H}}$. All the results presented in this paper, as well as their derivation, essentially remain the same in the case of $m = \infty$. However, in order to avoid some technicalities, we assume that $m < \infty$ or equivalently, that $\bar{\mathcal{H}}$ is finite dimensional.

Remark 3 The time evolution of κ defined by (19) is obviously not the most general. A straightforward extension of the present framework would be to introduce jumps in the dynamics of κ. This is natural since we do allow for jumps in the underlying process. And, although some of the derivations in the subsequent sections will have to be modified if κ has jumps, we believe that there is no serious obstacles for treating this case. However, we restrict our framework to the continuous evolution of the code-book, in order to increase the transparency of the results and their derivations.

We can now give the definition of a *tangent Lévy model*.

Definition 4 A pair of stochastic processes $(S_t,\kappa_t)_{t\in[0,\bar{T}]}$, where S is a positive (scalar) martingale and κ is a dynamic Lévy density, form a **tangent Lévy (tL) model** if, for any $x\in\mathbb{R}$, $T\in(0,\bar{T}]$ and $t\in[0,T)$, the following equality holds almost surely

$$C_t^{S_t,\kappa_t}(T,x)=\mathbb{E}\left(\left(S_T-K\right)^+\Big|\,\mathcal{F}_t\right),$$

where $C_t^{S_t,\kappa_t}(T,x)$ is defined by (7), for each (t,ω), using $\kappa_{t,\omega}(\,\cdot\,,\,\cdot\,)$ in lieu of $\kappa(\,\cdot\,,\,\cdot\,)$.

Notice that (17) implies that S is a local martingale. However, the martingale property does not follow immediately and has to be enforced exogenously, by, for example, assuming a form of *Novikov condition* for pure jump processes.

Remark 5 The martingale property of S can be guaranteed by the following version of Novikov condition

$$\mathbb{E}\exp\left(\frac{e}{2}\int_0^{\bar{T}}\|K_t\|_{\mathcal{B}^0}dt\right)<\infty.$$

This follows from Theorem IV.6 in [27] and the following estimate

$$\left|xe^x-e^x+1\right|\le\frac{e}{2}\left(|x|\wedge1\right)|x|(e^x+1),$$

which holds for all $x\in\mathbb{R}$.

Another way to ensure the martingale property is presented in Section 5.

Finally, for the sake of simplicity, we make some regularity assumptions on the structure of $\beta_t^n(T,x)$ as a function of x. These assumptions will only be used at the end of the proof of Theorem 12, namely, to compute the right hand side of (30). Roughly speaking, the regularity assumptions make sure that the derivatives of $\beta_t^n(T,\,\cdot\,)$ are well defined, decay exponentially at infinity and satisfy locally some integrability properties.

For convenience, we introduce

$$I_{t,\varepsilon}^{n,k}:=\sup_{T\in[t,\bar{T}]}\left[\operatorname{esssup}_{x\in\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(e^x+1)\left|\partial_{x^k}^k\beta_t^n(T,x)\right|\right.$$
$$\left.+\int_{\mathbb{R}}(e^x+1)|x|^3\left(|x|\wedge1\right)^{k-1}\left|\partial_{x^k}^k\beta_t^n(T,x)\right|dx\right],$$

whenever the derivatives appearing in right hand side are well defined.

Regularity Assumptions. For each $n\le m$, almost surely, for almost every $t\in[0,\bar{T}]$, we have:

RA1 For every $T \in [t, \bar{T}]$, the function $\beta_t^n(T, \cdot)$ is continuously differentiable on $\mathbb{R} \setminus \{0\}$, and its derivative is absolutely continuous.
RA2 For any $\varepsilon > 0$, $\sum_{k=0}^{2} I_{t,\varepsilon}^{n,k} < \infty$.

The above assumptions can be relaxed, if we decrease the order of singularity of $\beta_t^n(T, \cdot)$ at zero. Namely, we obtain

Alternative Regularity Assumptions. For each $n \leq m$, almost surely, for almost every $t \in [0, \bar{T}]$, we have:

ARA1 $\sup_{T\in[t,\bar{T}]} \int_{-1}^{1} |x| \, |\beta_t^n(T, x)| \, dx < \infty$
ARA2 For every $T \in [t, \bar{T}]$, the function $\beta_t^n(T, \cdot)$ is absolutely continuous on $\mathbb{R} \setminus \{0\}$.
ARA3 For any $\varepsilon > 0$, $\sum_{k=0}^{1} I_{t,\varepsilon}^{n,k} < \infty$.

These alternative regularity assumptions are used in Corollary 13 in order to simplify the "drift restriction" in Theorem 12. The improved "drift restriction" is used in Section 5.

4 Consistency Conditions

The main objective of this section is to provide necessary and sufficient conditions for a given underlying process and a dynamic Lévy density to form a tangent Lévy model. These conditions are expressed explicitly in terms of the semimartingale characteristics of these processes. These *consistency conditions* are stated in Theorem 12.

The notation of Section 3 holds throughout. In particular, throughout this section, unless otherwise specified, $S = (S_t)_{t\in[0,\bar{T}]}$ is a cádlág martingale, satisfying (17), with the corresponding random measure M and its compensator K (described in Section 3), and $\kappa = (\kappa_t)_{t\in[0,\bar{T}]}$ always stands for a *dynamic Lévy density*, with corresponding Brownian motion B and processes α and β (as described in Definition 1). Some of the formulas from Section 2 (namely, (7) and (13)) are also used in this section, with $\kappa_{t,\omega}(\cdot,\cdot)$ in lieu of $\kappa(\cdot,\cdot)$. We begin with

Proposition 6 *A cádlág martingale* $(S_t)_{t\in[0,\bar{T}]}$ *and a dynamic Lévy density* $(\kappa_t)_{t\in[0,\bar{T}]}$ *form a tangent Lévy model if and only if, for any* $x \in \mathbb{R}$ *and* $T \in (0, \bar{T}]$*, the call price process* $\left(C_t^{S_t,\kappa_t}(T, x)\right)_{t\in[0,T)}$ *produced by* κ *is a martingale.*

Proof:

The fact that the martingale property is necessary follows immediately from the definition of a tL model. So we only prove sufficiency. Fix some $T \in (0, \bar{T}]$ and notice that every call price $C_t^{S_t,\kappa_t}(T, x)$, defined via (7), is bounded by S_t, which implies that the call price process is uniformly integrable. The martingale convergence theorem yields that, as $t \nearrow T$, each call price process has a limit, in "almost sure" and $\mathbb{L}^1(\Omega)$ sense, and we show that this limit is $(S_{T-} - e^x)^+$. First, notice that $\|\kappa_t(T, \cdot)\|_{\mathcal{B}^0}$ is almost surely bounded over $t \in [0, T]$ and make use of the estimate (20) to conclude that, as $t \nearrow T$

$$\exp\left(-2\pi(2\pi\xi^2 + i\xi) \int_{T\wedge t}^{T} \hat{\psi}(\kappa_t(u); \xi) du\right) \to 1,$$

for all $\xi \in \mathbb{R}$. This yields that, as $t \nearrow T$, $\hat{\Delta}_t(T, \xi)$ given by (13) converges to

$$\frac{e^{\log S_{T-}(1-2\pi i\xi)}}{(1 - 2\pi i\xi)}$$

in $\mathbb{L}^2(\mathbb{R})$, as a function of ξ. Since the Fourier transform is unitary on $\mathbb{L}^2(\mathbb{R})$, we conclude that $\Delta_t(T,x)$ converges in $\mathbb{L}^2(\mathbb{R})$, as a function of x, to $e^x \mathbf{1}_{(-\infty,\log S_{T-}]}(x)$. Therefore, there is a sequence $t_n \nearrow T$, such that $\Delta_{t_n}(T,x)$ converges (to the same limit) for almost every $x \in \mathbb{R}$. Now, recall (7) and apply the dominated convergence theorem to conclude that almost surely, the call prices vanish, as x goes to infinity. This, together with the nonnegativity of $\Delta_t(T,x)$, implies that

$$C_t^{S_t,\kappa_t}(T,x) = \int_x^\infty \Delta_t(T,y)dy.$$

From the convergence of the call prices, we conclude that the above integral converges almost surely along $\{t_n\}$. Recall that the $\mathbb{L}^1([x,\infty))$ and "almost everywhere" limits of $\Delta_{t_n}(T,\,\cdot\,)$ should coincide, which gives us the desired expression for the limit of call prices. It only remains to notice that $S_{T-} = S_T$ almost surely, since S does not have any *fixed points of jump*, because of the absolute continuity of its compensator. ∎

Thus, in order to characterize consistency of S and κ, we need to determine when the call prices produced by κ are martingales. It may seem reasonable to pursue the following strategy: consider the (T,x)-surface of call prices at time t as a function of S_t and κ_t, prove Fréchet differentiability of this function, then apply an infinite dimensional version of Itô's formula to obtain the semimartingale representation of call prices, and, finally, set the drift term to zero. This approach was successfully used in [3]. However, Fréchet differentiability of the call prices with respect to κ cannot be proven by direct computation in the present situation: in particular, straightforward differentiation inside the integral in (16) results in a non-integrable expression. To take full advantage of the specifics of our set-up, we characterize the martingale property of call prices in the Fourier domain first, and then "carry it through" by Fourier inversion.

4.1 Semimartingale Property in Fourier Domain

First, we need to show that $\hat{\Delta}_t(T,\xi)$ defined by (13), with $\kappa_t(\,\cdot\,,\,\cdot\,)$ in lieu of $\kappa(\,\cdot\,,\,\cdot\,)$, is a semimartingale as a process in t. Fix any $T \in (0,\bar{T}]$, $\xi \in \mathbb{R}$ and $\varepsilon \in (0,T)$ and consider the mapping

$$\mathbf{F}_{T,\xi} : \mathcal{B} \times [0, T-\varepsilon] \hookrightarrow \mathbb{R},$$

given by

$$\mathbf{F}_{T,\xi}(v,t) = \exp\left(-2\pi(2\pi\xi^2 + i\xi) \int_{t\wedge T}^{T} \hat{\psi}(v(u);\xi)du \right),$$

where $\hat{\psi}$ is defined in (11). Next we study the properties of $\mathbf{F}_{T,\xi}(\,\cdot\,,\,\cdot\,)$.

Proposition 7 *1. For each $v \in \mathcal{B}$, $\mathbf{F}_{T,\xi}(v,\,\cdot\,)$ is continuously differentiable on $[0,T-\varepsilon]$, and the partial derivative $\partial \mathbf{F}_{T,\xi}/\partial t$ is jointly continuous on $\mathcal{B} \times [0,T-\varepsilon]$.*

2. *For each $t \in [0, T-\varepsilon]$, $\mathbf{F}_{T,\xi}(\,\cdot\,,t)$ is twice continuously Fréchet differentiable, and for any $h, h' \in \mathcal{B}$ we have*

$$\mathbf{F}'_{T,\xi}(v,t)[h] = -2\pi(2\pi\xi^2 + i\xi)\int_t^T \hat{\psi}(h(u);\xi)du \cdot \exp\left(-2\pi(2\pi\xi^2+i\xi)\int_t^T \hat{\psi}\left(v(u);\xi\right)du\right),$$

$$\mathbf{F}''_{T,\xi}(v,t)[h,h'] = 4\pi^2(2\pi\xi^2+i\xi)^2\int_t^T \hat{\psi}\left(h(u);\xi\right)du \int_t^T \hat{\psi}\left(h'(u);\xi\right)du \cdot \exp\left(-2\pi(2\pi\xi^2+i\xi)\int_t^T \hat{\psi}\left(v(u);\xi\right)du\right).$$

Proof:
Since we limit ourselves to $t < T-\varepsilon$, it is clear that:

$$\frac{\partial}{\partial t}\mathbf{F}_{T,\xi}(v,t) = 2\pi(2\pi\xi^2+i\xi)\hat{\psi}(v(t);\xi)\exp\left(-2\pi(2\pi\xi^2+i\xi)\int_t^T \hat{\psi}(v(u);\xi)du\right).$$

Notice that ψ can be viewed as a continuous linear operator from $\mathcal{B}^0$ into $\mathbb{L}^1(\mathbb{R})$, since

$$\int_{\mathbb{R}} |\psi(f;x)|\,dx \le c_1\int_{\mathbb{R}} (|x|\wedge 1)\,|x|(e^x+1)\,|f(x)|\,dx, \tag{20}$$

where c_i's, appearing here and further in the paper, are positive constants. The above implies that $\hat{\psi}$ is a continuous operator from $\mathcal{B}^0$ into $C(\mathbb{R})$. Then we have

$$\begin{aligned}
&\|\hat{\psi}(v_1(t_1)) - \hat{\psi}(v_2(t_2))\|_{C(\mathbb{R})} \\
&\quad\le \|\hat{\psi}(v_1(t_1)-v_1(t_2))\|_{C(\mathbb{R})} + \|\hat{\psi}(v_1(t_2)-v_2(t_2))\|_{C(\mathbb{R})} \\
&\quad\le \|\hat{\psi}\|_{\mathcal{B}^0\hookrightarrow C(\mathbb{R})}\int_{t_1\wedge t_2}^{t_1\vee t_2}\left\|\frac{d}{du}v_1(u)\right\|_{\mathcal{B}^0}du + \|\hat{\psi}\|_{\mathcal{B}^0\hookrightarrow C(\mathbb{R})}\|v_1(t_2)-v_2(t_2)\|_{\mathcal{B}^0} \\
&\quad\le \|\hat{\psi}\|_{\mathcal{B}^0\hookrightarrow C(\mathbb{R})}\left(\int_{t_1\wedge t_2}^{t_1\vee t_2}\left\|\frac{d}{du}v_1(u)\right\|_{\mathcal{B}^0}du + \|v_1-v_2\|_{\mathcal{B}}\right).
\end{aligned}\tag{21}$$

Using the above inequality, it is easy to see that $\frac{\partial}{\partial_t}\mathbf{F}_{T,\xi}(\,\cdot\,,\,\cdot\,)$ is jointly continuous.

Expressions for the first two derivatives of $\mathbf{F}_{T,\xi}$ with respect to v follow immediately from (20) and the estimates on residuals in the Taylor expansion of the exponential function. Their continuity follows, again, from the estimate (21). ∎

Corollary 8 *The stochastic process $\left\{\mathbf{F}_{T,\xi}(\kappa_t,t)\right\}_{t\in[0,T-\varepsilon]}$ is an adapted continuous semi-martingale with the following decomposition*

$$\mathbf{F}_{T,\xi}(\kappa_t,t) = \mathbf{F}_{T,\xi}(\kappa_0,0) + \int_0^t\left(\frac{\partial}{\partial u}\mathbf{F}_{T,\xi}(\kappa_u,u) + \mathbf{F}'_{T,\xi}(\kappa_u,u)[\alpha_u] + \frac{1}{2}\sum_{n=1}^m \mathbf{F}''_{T,\xi}(\kappa_u,u)[\beta_u^n,\beta_u^n]\right)du + \sum_{n=1}^m\int_0^t \mathbf{F}'_{T,\xi}(\kappa_u,u)[\beta_u^n]dB_u^n.$$

Proof:

Follows immediately from Itô's lemma for conditional Banach spaces (see, for example, Theorem III.5.4 in [25]). ∎

Corollary 9 *The stochastic process* $\left(\hat{\Delta}_t(T,\xi) = \frac{e^{\log S_t(1-2\pi i\xi)}}{1-2\pi i\xi}\mathbf{F}_{T,\xi}(\kappa_t,t)\right)_{t\in[0,T-\varepsilon]}$ *is an adapted semimartingale with the following decomposition*

$$\begin{aligned}
&\hat{\Delta}_t(T,\xi) = \hat{\Delta}_0(T,\xi) + \\
&\int_0^t \frac{e^{\log S_u(1-2\pi i\xi)}}{1-2\pi i\xi}\left[\frac{\partial}{\partial u}\mathbf{F}_{T,\xi}(\kappa_u,u) + \mathbf{F}'_{T,\xi}(\kappa_u,u)[\alpha_u] + \frac{1}{2}\sum_{n=1}^{m}\mathbf{F}''_{T,\xi}(\kappa_u,u)[\beta_u^n,\beta_u^n]\right.\\
&\left.+\int_{\mathbb{R}}\mathbf{F}_{T,\xi}(\kappa_u,u)\left(e^{x(1-2\pi i\xi)} - e^x(1-2\pi i\xi) - 2\pi i\xi\right)K_u(x)dx\right]du\\
&\quad+\sum_{n=1}^{m}\int_0^t \frac{e^{\log S_{u-}(1-2\pi i\xi)}}{1-2\pi i\xi}\mathbf{F}'_{T,\xi}(\kappa_u,u)[\beta_u^n]dB_u^n\\
&\quad+\int_0^t\int_{\mathbb{R}}\frac{e^{\log S_{u-}(1-2\pi i\xi)}}{1-2\pi i\xi}\mathbf{F}_{T,\xi}(\kappa_u,u)(e^{x(1-2\pi i\xi)}-1)[M(dx,du) - K_u(x)dxdu]
\end{aligned}$$

Proof:

Follows from the previous corollary and the general form of Ito's lemma applied to semimartingales with jumps (see, for example, Theorem I.4.57 in [23]). ∎

Notice that the values of $\mathbf{F}_{T,\xi}$ and its derivatives do not depend upon ε, only the "time" domain does. Then, since we can choose $\varepsilon > 0$ arbitrarily small, the semimartingale decomposition given in Corollary 9 holds for all $t \in [0,T)$, and we can drop ε.

Still for T and ξ fixed, we introduce the processes

$$\left(\mu_t(T,\xi), \left\{\nu_t^n(T,\xi)\right\}_{n=1}^m, j_t(T,\xi)\right)_{t\in[0,T)}$$

defined by

$$\begin{aligned}
\mu_t(T,\xi) &= \frac{e^{\log S_t(1-2\pi i\xi)}}{1-2\pi i\xi}\left[\frac{\partial}{\partial_t}\mathbf{F}_{T,\xi}(\kappa_t,t) + \mathbf{F}'_{T,\xi}(\kappa_t,t)[\alpha_t] + \frac{1}{2}\sum_{n=1}^{m}\mathbf{F}''_{T,\xi}(\kappa_t,t)[\beta_t^n,\beta_t^n]\right.\\
&\qquad\left.+\mathbf{F}_{T,\xi}(\kappa_t,t)\int_{\mathbb{R}}\left(e^{x(1-2\pi i\xi)} - e^x(1-2\pi i\xi) - 2\pi i\xi\right)K_t(x)dx\right],\\
\nu_t^n(T,\xi) &= \frac{e^{\log S_{t-}(1-2\pi i\xi)}}{1-2\pi i\xi}\mathbf{F}'_{T,\xi}(\kappa_t,t)[\beta_t^n],\\
j_t(T,\xi) &= \frac{e^{\log S_{t-}(1-2\pi i\xi)}}{1-2\pi i\xi}\mathbf{F}_{T,\xi}(\kappa_t,t),
\end{aligned}$$

so that

$$\begin{aligned}
\hat{\Delta}_t(T,\xi) = \hat{\Delta}_0(T,\xi) &+ \int_0^t \mu_u(T,\xi)du + \sum_{n=1}^{m}\int_0^t \nu_u^n(T,\xi)dB_u^n\\
&+\int_0^t\int_{\mathbb{R}} j_u(T,\xi)(e^{x(1-2\pi i\xi)}-1)\left[M(dx,du) - K_u(x)dxdu\right].
\end{aligned}$$

4.2 Main Result

In order to go back from the Fourier domain to the space domain, we need to use the inverse Fourier transform of *generalized functions* or *Schwartz distributions*, and consequently, we need to understand, as we start varying ξ, in which spaces the above stochastic processes take values.

We denote by $\mathcal{S}^0$ the space of bounded Borel functions on $\mathbb{R}$ which decay at infinity faster than any negative power of $|x|$.

Proposition 10 *For any $\phi \in \mathcal{S}^0$, $T \in (0, \bar{T}]$ and $t \in [0, T)$, we have, almost surely:*

$$\int_0^t \int_{\mathbb{R}} |\mu_u(T,\xi)|\,|\phi(\xi)| d\xi du < \infty,$$
$$\int_0^t \int_{\mathbb{R}} \left(\nu_u^n(T,\xi)\right)^2 \phi^2(\xi) d\xi du < \infty, \quad n = 1, \ldots, m$$
$$\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} j_u^2(T,\xi) \left(e^{x(1-2\pi i\xi)} - 1\right)^2 \phi^2(\xi) d\xi M(dx, du) < \infty.$$

Proof:

Recall that (15) yields

$$\left|\mathbf{F}_{T,\xi}(\kappa_t, t)\right| \le c_1.$$

Similarly, we have

$$\left|\frac{\partial}{\partial t}\mathbf{F}_{T,\xi}(\kappa_t, t)\right| \le c_2(1 + |\xi|^2)\|\kappa_t\|_{\mathcal{B}},$$
$$\left|\mathbf{F}'_{T,\xi}(\kappa_t, t)[h_t]\right| \le c_3(1 + |\xi|^2)\|h_t\|_{\mathcal{B}},$$
$$\left|\mathbf{F}''_{T,\xi}(\kappa_t, t)[h_t, h_t]\right| \le c_4(1 + |\xi|^4)\|h_t\|^2_{\mathcal{B}},$$

and also

$$\int_{\mathbb{R}} \left|e^{x(1-2\pi i\xi)} - e^x(1 - 2\pi i\xi) - 2\pi i\xi\right| K_t(x) dx$$
$$\le c_5(1 + |\xi|^2) \int_{\mathbb{R}} (|x| \wedge 1)^2 \,(e^x + 1) K_t(x) dx \le c_5(1 + |\xi|^2)\|K_t\|_{\mathcal{B}^0}.$$

Therefore

$$|\mu_t(T,\xi)| \le c_6 S_t (1 + |\xi|^3) \left(\|\kappa_t\|_{\mathcal{B}} + \|K_t\|_{\mathcal{B}^0} + \|\alpha_t\|_{\mathcal{B}} + \sum_{n=1}^m \|\beta_t^n\|^2_{\mathcal{B}}\right),$$
$$\left|\nu_t^n(T,\xi)\right| \le c_7 S_{t-}\,(1 + |\xi|)\,\|\beta_t^n\|_{\mathcal{B}}.$$

And since we have, almost surely

$$\sup_{t \in [0,\bar{T}]} (S_t + \|\kappa_t\|_{\mathcal{B}}) < \infty,$$

by construction, the integrability properties of α, β^n's and K, the definition of $\mathcal{S}^0$, together with the above estimates imply the first two inequalities of the proposition.

In order to prove the remaining inequality, we recall that, as discussed in Section 3, $M(dx, du)$ has only a finite number of atoms in $(\mathbb{R} \setminus [-1, 1]) \times [0, t]$ and, hence, it is enough to show that

$$\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} j_u^2(T, \xi) \left(e^{x(1-2\pi i\xi)} - 1\right)^2 \mathbf{1}_{[-1,1]}(x)\phi^2(\xi) d\xi M(dx, du) < \infty. \tag{22}$$

holds almost surely. Since

$$j_t^2(T, \xi) \left(e^{x(1-2\pi i\xi)} - 1\right)^2 \mathbf{1}_{[-1,1]}(x) \leq c_8 S_{t-}^2 \left(|x| \wedge 1\right)^2, \tag{23}$$

the left hand side of (22) is finite almost surely, as it is bounded from above by

$$c_9 \sup_{u\in[0,\bar{T}]} \left(S_u^2\right) \int_0^t \int_{\mathbb{R}} \left(|x| \wedge 1\right)^2 M(dx, du) < \infty.$$

Notice that the nonnegativity of κ_t is required in order to make use of (15), which only makes sense if $T^{-1} \int_0^T \kappa_t(u, \cdot)du$ can serve as a Lévy density. ∎

We use the standard notation $\mathcal{S}$ for the *Schwartz space* of (complex-valued) C^∞ functions on $\mathbb{R}$ whose derivatives of all orders decay at infinity faster than any negative power of $|x|$. Then any polynomially bounded Borel function f can be viewed as a continuous functional on $\mathcal{S}$ via the duality

$$\langle f, \phi\rangle = \int_{\mathbb{R}} f(x)\overline{\phi(x)}dx. \tag{24}$$

Corollary 11 *For any $\phi \in \mathcal{S}$, $T \in (0, \bar{T}]$ and $t \in [0, T)$, the following equality holds almost surely:*

$$\langle \hat{\Delta}_t(T, \cdot), \phi\rangle = \langle \hat{\Delta}_0(T, \cdot), \phi\rangle + \int_0^t \langle \mu_u(T, \cdot), \phi\rangle du + \sum_{n=1}^m \int_0^t \langle \nu_u^n(T, \cdot), \phi\rangle dB_u^n$$
$$+ \int_0^t \int_{\mathbb{R}} \langle j_u(T, \cdot)(e^{x(1-2\pi i \cdot)} - 1), \phi\rangle \left[M(dx, du) - K_u(x)dxdu\right]$$

Proof:

We use Fubini's theorem to change the order of integration in the first integral, and the absolute integrability follows from Proposition 10.

Changing the order of integration in the last two integrals can be justified by the stochastic Fubini's theorem (see, for example, Theorem 65 in [30]), which requires integrability of the square of the integrand with respect to "$d\xi \times d$[quadratic variation of the stochastic integrator]". This is justified, again, by Proposition 10. ∎

Finally, we formulate the *consistency conditions*, namely, the necessary and sufficient conditions for the pair (S, κ) to form a tangent Lévy model (see Definition 4), expressed in terms of their semimartingale characteristics.

Theorem 12 *Under the regularity assumptions RA1-RA2 of Section 3, a cádlág martingale $(S_t)_{t\in[0,\bar{T}]}$, satisfying (17), and a dynamic Lévy density $(\kappa_t)_{t\in[0,\bar{T}]}$ form a tangent Lévy model if and only if the following conditions hold almost surely for almost every $x \in \mathbb{R}$ and $t \in [0, \bar{T})$, and all $T \in (t, \bar{T}]$:*

– *Drift restriction*

$$\alpha_t(T,x) = -e^{-x}\cdot\sum_{n=1}^{m}\int_{\mathbb{R}}\partial^4_{y^4}\psi\left(\bar\beta^n_t(T);y\right)\left[\psi\left(\beta^n_t(T);x-y\right)-\right.$$
$$\left.\left(1-y\partial_x+\frac{y^2}{2}\partial^2_{x^2}-\frac{y^3}{6}\partial^3_{x^3}\right)\psi\left(\beta^n_t(T);x\right)\right]-$$
$$\partial^2_{y^2}\psi\left(\bar\beta^n_t(T);y\right)\left[\psi\left(\beta^n_t(T);x-y\right)-(1-y\partial_x)\,\psi\left(\beta^n_t(T);x\right)\right]dy, \tag{25}$$

– *Compensator specification*

$$K_t(x)=\kappa_t(t,x). \tag{26}$$

We use the notation

$$\bar\beta^n_t(T)=\int_{t\wedge T}^{T}\beta^n_t(u)du,$$

and we understand functions of the form $\psi\left(f;\,\cdot\,\right)$, and their derivatives, as defined separately on $(0,\infty)$ and $(-\infty,0)$.

Proof:

In view of Proposition 6, it is enough to show that equations (25) and (26) hold if and only if all the call prices, produced by κ, are martingales (up until expiry). Recall that the Fourier transform is a bijection on $\mathcal{S}$, and it is defined on the space $\mathcal{S}^*$ of tempered distributions (i.e. the topological dual of $\mathcal{S}$) via the duality (24). So, viewing $\hat\Delta_t(T,\,\cdot\,)$ as an element of $\mathcal{S}^*$, we have:

$$\langle\hat\Delta_t(T,\,\cdot\,),\phi\rangle=\langle\Delta_t(T,\,\cdot\,),\hat\phi\rangle,$$

and therefore, for any $\phi\in\mathcal{S}$, Corollary 11 yields

$$\langle\Delta_t(T,\,\cdot\,),\phi\rangle=\langle\hat\Delta_t(T,\,\cdot\,),\check\phi\rangle=\langle\Delta_0(T,\,\cdot\,),\phi\rangle+\int_0^t\langle\mu_u(T,\,\cdot\,),\check\phi\rangle du$$
$$+\sum_{n=1}^{m}\int_0^t\langle\nu^n_u(T,\,\cdot\,),\check\phi\rangle dB^n_u \tag{27}$$
$$+\int_0^t\int_{\mathbb{R}}\langle j_u(T,\,\cdot\,)(e^{x(1-2\pi i\,\cdot\,)}-1),\check\phi\rangle\left[M(dx,du)-K_u(x)dxdu\right].$$

We now show that the martingale property of the call prices produced by κ is equivalent to: *almost surely for almost all* $t\in[0,\bar T)$*,* $\mu_t(T,\xi)=0$ *for all* $T\in(t,\bar T]$ *and all* $\xi\in\mathbb{R}$, or, in other words, $\mu\equiv0$. Notice that almost surely for all $t\in[0,\bar T)$, the function $\{\mu_t(T,\xi)\}_{T\in(t,\bar T],\xi\in\mathbb{R}}$ is jointly continuous. This observation is not necessary for the proof but helps avoid ambiguity in understanding what it means for $\mu_t(\,\cdot\,,\,\cdot\,)$ to be equal to zero.

If $\mu\equiv0$, we choose a sequence $\left\{\phi^k\right\}_{k=1}^{\infty}$ in $\mathcal{S}$, such that

$$\phi^k(x)\downarrow\mathbf{1}_{[a,b]}(x),$$

for every $x\in\mathbb{R}$. This sequence, of course, will also converge in $\mathbb{L}^1(\mathbb{R})$. Making use of (27), we conclude that

$$\left\{\left(\langle\Delta_t(T,\,\cdot\,),\phi^k\rangle\right)_{t\in[0,T)}\right\}_{k=1}^{\infty} \tag{28}$$

is a sequence of local martingales. Since each of them is bounded by a constant times S_t, it is in fact a sequence of true martingales. The limit as $k \to \infty$ of this sequence is, almost surely, for any $t \in [0,T)$, equal to

$$\langle \Delta_t(T,\cdot), \mathbf{1}_{[a,b]} \rangle = C_t^{S_t,\kappa_t}(T,a) - C_t^{S_t,\kappa_t}(T,b).$$

Since (28) is an almost surely decreasing sequence of martingales, by monotone convergence, its limit is a martingale. Thus, for any $a, b \in \mathbb{R}$, the difference

$$\left(C_t^{S_t,\kappa_t}(T,a) - C_t^{S_t,\kappa_t}(T,b) \right)_{t\in[0,T)}$$

is a martingale. Finally, since call prices almost surely decrease to zero, as strike goes to infinity, applying monotone convergence again, we conclude that all the call prices are martingales.

Conversely, if all the call prices produced by κ are martingales, then for any $\phi \in \mathcal{S}$ we have that $\left(\langle C_t^{S_t,\kappa_t}(T,\cdot), \phi \rangle \right)_{t\in[0,T)}$ is a martingale as well. To see this, recall that a call price is a continuous function of log-strike and it is bounded by S_t. Then $\langle C_t^{S_t,\kappa_t}(T,\cdot), \phi \rangle$ can be viewed as a limit of Riemann sums $X_t^n(T)$, where the limit is understood for each $t \in [0,T)$ in "almost sure" sense. Varying t we find that each $X_\cdot^n(T)$ is a martingale. From the dominated convergence theorem then, we see that $X_t^n(T)$ converges to $\langle C_t^{S_t,\kappa_t}(T,\cdot), \phi \rangle$ in $\mathbb{L}^1(\Omega)$, and therefore, the limit is also a martingale.

For any $\phi \in \mathcal{S}$,

$$\langle \Delta_t(T,\cdot), \phi \rangle = \langle C_t^{S_t,\kappa_t}(T,\cdot), \phi' \rangle$$

is also a martingale since $\phi' \in \mathcal{S}$. Due to (27), this implies that for any $\phi \in \mathcal{S}$ and any $T \in (0,\bar{T}]$, almost surely for almost all $t \in [0,T)$, we have

$$\langle \mu_t(T,\cdot), \phi \rangle = 0.$$

Now, we can choose a dense countable subset of $\mathcal{S}$ and conclude that, almost surely for almost all $t \in [0,\bar{T})$, the above equality holds for all rational $T \in (t,\bar{T}]$ and all functions ϕ from the chosen set. This implies $\mu \equiv 0$.

Thus, the martingale property of the call prices produced by κ is equivalent to $\mu \equiv 0$. Let us now formulate this condition in terms of α, β and K. Notice that an absolutely continuous function is equal to zero on an interval if and only if it is zero at a boundary point, and its derivative is zero almost everywhere in the interval. In order to simplify the analysis of the derivative, we will work with $\mu_t(T,\xi)/\mathbf{F}_{T,\xi}(\kappa_t,t)$ instead of $\mu_t(T,\xi)$ (clearly, $\mu_t(T,\xi) = 0$ if and only if $\mu_t(T,\xi)/\mathbf{F}_{T,\xi}(\kappa_t,t) = 0$). Letting $T \searrow t$ in the equation $\mu_t(T,\xi)/\mathbf{F}_{T,\xi}(\kappa_t,t) = 0$, we obtain

$$-2\pi(2\pi\xi^2 + i\xi)\hat{\psi}(\kappa_t(t);\xi) = \int_{\mathbb{R}} \left(e^{x(1-2\pi i\xi)} - e^x(1-2\pi i\xi) - 2\pi i\xi \right) K_t(x)dx$$

which is equivalent to (26). To see this, we use the derivations given in detail in Appendix A and conclude that the above right hand side is equal to $-2\pi(2\pi\xi^2 + i\xi)\hat{\psi}(K_t;\xi)$, which implies that $\hat{\psi}(\kappa_t(t) - K_t;\xi) = 0$, which is equivalent to (26).

Notice that the T-derivative of $\mu_t(T,\xi)/\mathbf{F}_{T,\xi}(\kappa_t,t)$ is well defined for all $T \in (t,\bar{T})$. Making use of the Proposition 7 and the definition of $\mu_t(T,x)$, we obtain

$$\partial_T \frac{\mu_t(T,\xi)}{\mathbf{F}_{T,\xi}(\kappa_t,t)} = \frac{e^{\log S_t(1-2\pi i\xi)}}{1-2\pi i\xi}\Bigg(-2\pi(2\pi\xi^2+i\xi)\hat{\psi}(\alpha_t(T);\xi) + 4\pi^2(2\pi\xi^2+i\xi)^2\sum_{n=1}^{m}\hat{\psi}\left(\beta_t^n(T);\xi\right)\int_t^T \hat{\psi}\left(\beta_t^n(u);\xi\right)du\Bigg)$$

Equating it to zero, we obtain

$$\hat{\psi}(\alpha_t(T);\xi) = 2\pi\left(2\pi\xi^2+i\xi\right)\sum_{n=1}^{m}\hat{\psi}\left(\beta_t^n(T);\xi\right)\hat{\psi}\left(\bar{\beta}_t^n(T);\xi\right).$$

Inverting the Fourier transform yields

$$\psi(\alpha_t(T);x) = -\sum_{n=1}^{m}\left[\partial^2_{x^2}+\partial_x\right]\left(\int_{\mathbb{R}}\psi\left(\beta_t^n(T);x-y\right)\psi\left(\bar{\beta}_t^n(T);y\right)dy\right), \quad (29)$$

where the derivatives are understood in a generalized sense (as operators on $\mathcal{S}^*$). The above implication follows immediately from the properties of the Fourier transform (understood in the generalized sense, acting on $\mathcal{S}^*$). It will be shown later that the derivatives in (29) exist in the classical sense. Assuming first that the right hand side of the above is well defined as a classical function, we solve (29) for α, or in other words, we invert the operator ψ. The inverse of ψ is $e^{-x}\left[\partial^2_{x^2}-\partial_x\right]$, which yields

$$\alpha_t(T,x) = -e^{-x}\sum_{n=1}^{m}\left[\partial^4_{x^4}-\partial^2_{x^2}\right]\left(\int_{\mathbb{R}}\psi\left(\beta_t^n(T);x-y\right)\psi\left(\bar{\beta}_t^n(T);y\right)dy\right), \quad (30)$$

given that the right hand side is well defined.

As mentioned above, the integral in (30) is well defined for all $x \neq 0$. However, a modicum of care is required differentiating it, since derivatives of the integrands are not absolutely integrable around zero. Typically, we need to compute an expression of the form

$$\partial_x \int_{\mathbb{R}} f(x-y)g(y)dy,$$

when $f,g \in \mathbb{L}^1(\mathbb{R})$ are both absolutely continuous outside any neighborhood of zero and vanish at infinity. We can also assume that their first derivatives are bounded and absolutely integrable outside any neighborhood of zero and, if multiplied by $|x|$, are locally absolutely integrable at zero. We should think of f and g as $\psi\left(\beta_t^n(T)\right)$ and $\psi\left(\bar{\beta}_t^n(T)\right)$ respectively. We use integration by parts to be able to pass the derivative under the integral. Without any loss of generality we assume that $x > 0$. Then

$$\begin{aligned}\int_{\mathbb{R}} f(x-y)g(y)dy = \lim_{\varepsilon\to 0}\Bigg[&\int_{-\infty}^{-\varepsilon}\partial_y\left(\int_{x-y}^{x+\varepsilon}f(z)dz\right)g(y)dy \\ &+\int_{\varepsilon}^{x}\partial_y\left(\int_{x-y}^{x-\varepsilon}f(z)dz\right)g(y)dy+\int_{x}^{\infty}\partial_y\left(\int_{x-y}^{x-\varepsilon}f(z)dz\right)g(y)dy\Bigg] \\ &= \int_{-\infty}^{x}g'(y)\int_{x}^{x-y}f(z)dzdy+\int_{x}^{\infty}g'(y)\int_{x}^{x-y}f(z)dzdy,\end{aligned}$$

from which we conclude

$$\partial_x \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} g'(y)\left(f(x-y)-f(x)\right)dy.$$

Clearly, if in addition we assume that the first three derivatives of f and g vanish at infinity, the first four derivatives of f and g are essentially bounded outside any neighborhood of zero, and the following expressions

$$\left\{ \frac{|x|^k}{|x|\vee 1} f^{(k)}(x),\ \frac{|x|^k}{|x|\vee 1} g^{(k)}(x) \right\}_{k=1}^{4} \tag{31}$$

are absolutely integrable functions of $x \in \mathbb{R}$, then, repeating the above derivations, we obtain

$$\begin{aligned}
&\partial^2_{x^2} \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} g''(y)\left(f(x-y)-f(x)+yf'(x)\right)dy,\\
&\partial^3_{x^3} \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} g'''(y)\left(f(x-y)-f(x)+yf'(x)-\frac{y^2}{2}f''(x)\right)dy, \qquad (32)\\
&\partial^4_{x^4} \int_{\mathbb{R}} f(x-y)g(y)dy =\\
&\qquad\int_{\mathbb{R}} g^{(4)}(y)\left(f(x-y)-f(x)+yf'(x)-\frac{y^2}{2}f''(x)+\frac{y^3}{6}f'''(x)\right)dy.
\end{aligned}$$

Let us now continue with (30). Notice that, although the definition of ψ involves only one integral, the integrand there depends upon the limit of integration, so that, effectively, $\psi(f;x)$ is a double exponentially weighted integral of f (see [6]). However, its derivative is an integral operator:

$$\partial_x \psi\left(f;x\right) = -e^x \int_x^{\mathrm{sign}(x)\infty} f(y)dy. \tag{33}$$

The k-th order derivative of $\psi\left(f;\cdot\right)$, for any $k \geq 2$, can be obtained by a straightforward calculation, and it takes the form of the exponential, e^x, multiplied by a linear combination of the integral of f and its first $k-2$ derivatives.

The above implies that, due to the regularity assumptions we made on the functions $\beta^n_t(\cdot,\cdot)$ (see RA1-RA2 in Section 3), the functions $\psi\left(\beta^n_t(T);\cdot\right)$ and $\psi\left(\bar\beta^n_t(T);\cdot\right)$ have all the properties of f and g, introduced above.

Thus, the derivatives in (29) and (30) are well defined in the classical sense, and (30) and (32) yield (25). ∎

As explained in Section 5, the additional integrability assumption ARA1 in Section 3 is a very natural one, and, under this assumption, the drift restriction (25) can be simplified. Namely, we have

Corollary 13 *Under the alternative regularity assumptions ARA1-ARA3 of Section* 3*, a cádlág martingale* $(S_t)_{t\in[0,\bar T]}$*, satisfying (17), and a dynamic Lévy density* $(\kappa_t)_{t\in[0,\bar T]}$ *form a tangent Lévy model if and only if, almost surely for almost every* $x \in \mathbb{R}$ *and* $t \in [0,\bar T)$*, and all* $T \in (t,\bar T]$*, the compensator specification (26) is satisfied and the following modification of the drift restriction holds*

$$\begin{aligned}
&\alpha_t(T,x) =\\
&-e^{-x}\sum_{n=1}^m \int_{\mathbb{R}} \partial^3_{y^3}\psi\left(\bar\beta^n_t(T);y\right)\left[\partial_x\psi\left(\beta^n_t(T);x-y\right)-(1-y\partial_x)\,\partial_x\psi\left(\beta^n_t(T);x\right)\right]\\
&\qquad -\partial_y\psi\left(\bar\beta^n_t(T);y\right)\partial_x\psi\left(\beta^n_t(T);x-y\right)dy. \qquad (34)
\end{aligned}$$

The above drift restriction becomes even more attractive after noticing that, in this case, the drift is expressed in terms of $\partial_x \psi\left(\beta_t^n(T);x\right)$ and $\partial_x \psi\left(\bar{\beta}_t^n(T);x\right)$, and these functions are, essentially, the first integrals of $\beta_t^n(T,\,\cdot\,)$ and $\bar{\beta}_t^n(T,\,\cdot\,)$ respectively (see (33)).

Proof:

Let us rewrite the end of the proof of Theorem 12, starting with equation (30). First, notice that if β_t^n takes values in $\mathcal{H}$ and the alternative regularity assumptions ARA1-ARA3 hold, $\partial_x \psi\left(\beta_t^n;x\right)$ and $\partial_x \psi\left(\bar{\beta}_t^n;x\right)$ are absolutely integrable in x. Therefore, using integration by parts, we can pass two differential operators inside the integral in (30) and obtain

$$\alpha_t(T,x) = -e^{-x}\sum_{n=1}^{m}\left[\partial_{x^2}^2 - 1\right]\left(\int_{\mathbb{R}} \partial_x \psi\left(\beta_t^n(T);x-y\right)\partial_y \psi\left(\bar{\beta}_t^n(T);y\right)dy\right)$$

We then proceed as in the proof of Theorem 12, making use of (32), to derive (34). ∎

In fact, if we assume in addition that $\beta_t^n(T,\,\cdot\,)$ is locally integrable at zero, then the drift restriction can be further simplified to take its most convenient form (see (47) and (40)), which is used in Section 6.

5 Existence of Tangent Lévy Models

In Theorem 12 of Section 4 we described the tangent Lévy models in terms of the semimartingale characteristics of their components, S and κ. The question is now, how to *parameterize explicitly a large family of tL models*? We would like to identify the *free parameter* whose value can be specified exogenously and whose admissible values determine uniquely the tangent Lévy model. From Theorem 12 we see that β is a good candidate. In this section we show how to construct a consistent tL model from any admissible value of β. However, in order to do so, we loose some generality: we introduce specifications that effectively reduce the class of tL models described in Section 3, but, at the same time, make them more tractable and amenable to implementation, and allow us to prove the existence result.

5.1 Choosing the Right Functional Subspaces

We first introduce a convenient specification of κ. A crucial point of the setup of Section 3 is the assumption of nonnegativity of κ. We would like to construct its dynamics in such a way that the nonnegativity property is preserved automatically. The most straight forward way to preserve nonnegativity, is to stop the process before it becomes negative. Unfortunately, the set of all $f(\,\cdot\,,\cdot\,) \in \mathcal{B}$, whose essential infimum is negative, is dense in $\mathcal{B}$, which means that we cannot control the corresponding stopping time by choosing the right initial condition κ_0. This is a problem for both numerical implementation of the model, and for the further development of the theory, as one, eventually, would like to construct dynamics of κ in such a way that it never leaves the set of nonnegative functions without having to be stopped (see Proposition 18).

Thus, we narrow down the state space $\mathcal{B}$ by fixing the asymptotic behavior of its elements at $x \to \infty$ and at $x \to 0$, so that κ is always of the form

$$\kappa_t(T,x) = e^{-\lambda|x|}\left(|x| \wedge 1\right)^{-1-2\delta}\tilde{\kappa}_t(T,x), \tag{35}$$

for some fixed $\lambda > 1$ and $\delta \in (0,1)$ and a function $\tilde{\kappa}_t(T,\,\cdot\,)$ which belongs to $\tilde{\mathcal{B}}^0 := \tilde{C}(\mathbb{R})$, the subspace of $C(\mathbb{R})$ consisting of continuous functions with limits at $\pm\infty$, equipped with the standard "sup" norm. Clearly, such functions $\kappa_t(T,\,\cdot\,)$ are in $\mathcal{B}^0$. Thus, we can specify the time evolution of the dynamic Lévy density κ_t by modeling $\tilde{\kappa}_t$. For notational convenience, we introduce

$$\rho(x) = e^{-\lambda|x|} \left(|x| \wedge 1\right)^{-1-2\delta}. \tag{36}$$

From now on, we will use the notation "tilde" for the functions normalized by ρ. The motivation for such a choice comes from the *CGMY* model introduced in [5].

Remark 14 Notice that the above specification is not the only possible. For example, we could have chosen κ to be of the form

$$\kappa_t(T,x) = \begin{cases} e^{-\lambda^+|x|}\left(|x|^{-1-2\delta} \vee 1\right)\tilde{\kappa}_t^+(T,x) \;\; x > 0, \\ e^{-\lambda^-|x|}\left(|x|^{-1-2\delta} \vee 1\right)\tilde{\kappa}_t^-(T,x) \;\; x < 0, \end{cases}$$

which corresponds to modeling the intensities of positive and negative jumps separately. All the results obtained in this chapter can be extended to include the above specification, with the only difference that we would have to study the dynamics of two functions $\tilde{\kappa}^+$ and $\tilde{\kappa}^-$ instead of a single one. However, for notational convenience, we will restrict ourselves to specification (35).

In order to define the dynamics of $\tilde{\kappa}$, we need to describe the state space of its diffusion coefficient $\tilde{\beta}$. We would like to construct the dynamics of $\tilde{\kappa}$ so that the Corollary 13 could be applied to $\kappa = \rho\tilde{\kappa}$, therefore, we need the alternative regularity assumptions ARA1-ARA3 in Section 3 to be satisfied. Thus, we choose a Hilbert space $\mathcal{G}$ of absolutely continuous functions on $\mathbb{R}$, whose first derivatives are in $\mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^\infty(\mathbb{R})$, and for which the following inequality holds

$$\|f'\|_{\mathbb{L}^1(\mathbb{R})} + \|f'\|_{\mathbb{L}^\infty(\mathbb{R})} \le c\|f\|_{\mathcal{G}},$$

for some positive constant c. For example, $\mathcal{G}$ can be defined as the space of functions on $\mathbb{R}$, whose first derivatives vanish outside of some fixed compact, and whose second derivatives are square integrable.

However, it is not enough to require that $\tilde{\beta}_t^n(T,\,\cdot\,)$ takes values in $\mathcal{G}$. Recall that we need to construct the dynamics of $\tilde{\kappa}$ so that the drift restriction is satisfied for $\kappa = \rho\tilde{\kappa}$. Analyzing (25) or (34), we conclude that as $x \to 0$, the asymptotic behavior of each term in the sums in the right hand sides of these equations depends only on the singularity of $\beta_t^n(T,\,\cdot\,)$ and $\bar{\beta}_t^n(T,\,\cdot\,)$ at zero. If we assume a power-type behavior of $\beta_t^n(T,x)$, say, $|x|^{-\epsilon}$, around $x = 0$, computing the asymptotic behavior of the integrals in (25) or (34), we see that their rate of growth as $x \to 0$, is given by $|x|^{-2\epsilon+1}$ (see, for example (61) for similar calculations). This means that the drift restriction can, potentially, increase the singularity at zero if $\beta_t^n(T,\,\cdot\,)$ is not integrable at zero. Notice also that, on the other hand, when $\epsilon \le 1$, the order of singularity will be decreased by the drift restriction. We know that the order of singularity of $\alpha_t(T,x)$ at $x = 0$ should not exceed $|x|^{-1-2\delta}$, therefore, we need $\epsilon \le 1+\delta$, which means that we have to restrict ourselves to $\beta_t^n(T,x)$'s which grow at most like $|x|^{-1-\delta}$ at $x = 0$. Studying the drift restriction, we can also notice that it can potentially create some growth at $x \to \infty$ (although not of a very high order), if $\beta_t^n(T,\,\cdot\,)$'s do not vanish fast enough at infinity. The reader can consult the derivation of the estimates proven in Appendix B for more details.

Motivated by the above, and expecting, naturally, that $\beta_t = \rho\tilde{\beta}_t$, we then define the Hilbert space $\tilde{\mathcal{H}}^0$ by

$$\tilde{\mathcal{H}}^0 = \left\{ e^{-\lambda'|\,\cdot\,|} \left(|\,\cdot\,|^\delta \wedge 1 \right) f(\,\cdot\,) \;\middle|\; f \in \mathcal{G} \right\},$$

where $\lambda' > 0$ is some fixed real number. The inner product on $\tilde{\mathcal{H}}^0$ is inherited from $\mathcal{G}$. Namely, if we rewrite functions $f, g \in \tilde{\mathcal{H}}^0$ in the form $f(x) = e^{-\lambda'|x|} \left(|x|^\delta \wedge 1 \right) \tilde{f}(x)$ and $g(x) = e^{-\lambda'|x|} \left(|x|^\delta \wedge 1 \right) \tilde{g}(x)$ with $\tilde{f}, \tilde{g} \in \mathcal{G}$, then

$$\langle f, g \rangle_{\tilde{\mathcal{H}}^0} := \langle \tilde{f}, \tilde{g} \rangle_{\mathcal{G}}$$

The spaces $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{H}}$, of functions of two variables, are then constructed from $\tilde{\mathcal{B}}^0$ and $\tilde{\mathcal{H}}^0$ in the same way as $\mathcal{B}$ and $\mathcal{H}$ were constructed from $\mathcal{B}^0$ and $\mathcal{H}^0$ in Section 3, namely, using the norms:

$$\|f\|_{\tilde{\mathcal{B}}} := \|f(0)\|_{\tilde{\mathcal{B}}^0} + \int_0^{\bar{T}} \left\| \frac{d}{du} f(u) \right\|_{\tilde{\mathcal{B}}^0} du < \infty,$$

$$\|f\|^2_{\tilde{\mathcal{H}}} := \|f(0)\|^2_{\tilde{\mathcal{H}}^0} + \int_0^{\bar{T}} \left\| \frac{d}{du} f(u) \right\|^2_{\tilde{\mathcal{H}}^0} du < \infty.$$

Since the surface $\tilde{\kappa}_t(\,\cdot\,,\,\cdot\,)$ is continuous, it is convenient to introduce the following stopping time

$$\tilde{\tau}_0 = \inf \left\{ t \geq 0 : \inf_{T \in [t, \bar{T}], x \in \mathbb{R}} \tilde{\kappa}_t(T, x) \leq 0 \right\}, \tag{37}$$

and stop process $\tilde{\kappa}$ at $\tilde{\tau}_0$. Notice that $\inf_{T \in [t, \bar{T}], x \in \mathbb{R}} \tilde{\kappa}_t(T, x)$ is an adapted continuous process in t, hence $\tilde{\tau}_0$ is a predictable stopping time (see, for example, Proposition I.2.13 in [23]). Notice that $\tilde{\kappa}_{t \wedge \tilde{\tau}_0}(\,\cdot\,,\,\cdot\,)$ is almost surely nonnegative, and therefore, so is $\kappa_{t \wedge \tilde{\tau}_0}(\,\cdot\,,\,\cdot\,)$.

Thus, we construct the dynamic Lévy density $\kappa = (\kappa)_{t \in [0, \bar{T}]}$ in the form $\kappa_t = \rho \tilde{\kappa}_{t \wedge \tilde{\tau}_0}$, with

$$\tilde{\kappa}_t = \tilde{\kappa}_0 + \int_0^t \tilde{\alpha}_u du + \sum_{n=1}^m \int_0^t \tilde{\beta}^n_u dB^n_u, \tag{38}$$

where $B = \left(B^1, \ldots, B^m \right)$ is a multidimensional Brownian motion, $\tilde{\alpha}$ is a progressively measurable integrable random process with values in $\tilde{\mathcal{B}}$, and each $\tilde{\beta}^n$ is a progressively measurable square integrable random process with values in $\tilde{\mathcal{H}}$.

It is not hard to see that $\kappa = (\rho \tilde{\kappa}_{t \wedge \tilde{\tau}_0})_{t \in [0, \bar{T}]}$ with $\tilde{\kappa}$ defined by (38), is indeed a dynamic Lévy density in the sense of Definition 1, with

$$\begin{aligned} &\alpha_t(T, x) = \rho(x) \tilde{\alpha}_t(T, x) \mathbf{1}_{t \leq \tilde{\tau}_0}, \\ &\beta^n_t(T, x) = \rho(x) \tilde{\beta}^n_t(T, x) \mathbf{1}_{t \leq \tilde{\tau}_0}, \;\; n = 1, \ldots, m. \end{aligned} \tag{39}$$

Recall that we are only interested in dynamic Lévy densities which are consistent with the underlying (so that the two form a tL model). It is easy to check that the assumptions ARA1-ARA3 of Section 3 are satisfied for β defined by (39), and applying Corollary 13, we rewrite the consistency conditions in the new variables:

$$\tilde{\alpha}_t(T, x) \mathbf{1}_{t \leq \tilde{\tau}_0} = Q^{\tilde{\beta}_t}(T, x)\, \mathbf{1}_{t \leq \tilde{\tau}_0}, \;\; K_t(x) = \rho(x) \tilde{\kappa}_t(t, x), \tag{40}$$

where we introduced the notation

$$Q^{\tilde{\beta}_t}(T,x) = -\frac{e^{-x}}{\rho(x)} \cdot \tag{41}$$
$$\sum_{n=1}^{m} \int_{\mathbb{R}} \partial^3_{y^3}\psi\left(\rho\bar{\tilde{\beta}}^n_t(T);y\right)\left[\partial_x\psi\left(\rho\tilde{\beta}^n_t(T);x-y\right) - (1-y\partial_x)\,\partial_x\psi\left(\rho\tilde{\beta}^n_t(T);x\right)\right]$$
$$- \partial_y\psi\left(\rho\bar{\tilde{\beta}}^n_t(T);y\right)\partial_x\psi\left(\rho\tilde{\beta}^n_t(T);x-y\right)dy,$$

and

$$\bar{\tilde{\beta}}^n_t(T) = \int_{t\wedge T}^{T} \tilde{\beta}^n_t(u)du.$$

In this section we only use the "sufficiency" of the consistency conditions given in Corollary 13. Therefore, we assume that (40) holds almost surely for all $x \in \mathbb{R}$ and all $t,T \in [0,\bar{T}]$. Notice that for any admissible $\tilde{\beta}$, we can use $\tilde{\alpha}_t = Q^{\tilde{\beta}_t}$ to construct $\tilde{\kappa} = (\tilde{\kappa})_{t\in[0,\bar{T}]}$ via (38), and then stop it at $\tilde{\tau}_0$ (clearly, the stochastic differential of the stopped process will have the drift $Q^{\tilde{\beta}_t}\mathbf{1}_{t\le\tilde{\tau}_0}$ and the diffusion coefficient $\tilde{\beta}_t\mathbf{1}_{t\le\tilde{\tau}_0}$). Then the only remaining question is whether the process $\left(Q^{\tilde{\beta}_t}\right)_{t\in[0,\bar{T}]}$ is admissible (satisfies the properties assumed for $\tilde{\alpha}$). The following lemma gives a positive answer to this question.

Lemma 15 *For any vector of progressively measurable square integrable $\tilde{\mathcal{H}}$-valued stochastic processes, $\tilde{\beta} = \left\{\tilde{\beta}^n\right\}_{n=1}^{m}$, the process $\left(Q^{\tilde{\beta}_t}(\,\cdot\,,\,\cdot\,)\right)_{t\in[0,\bar{T}]}$, defined in (41), is a progressively measurable integrable random process with values in $\tilde{\mathcal{B}}$.*

Proof:

Given in Appendix B. ■

The above algorithm gives us the dynamic Lévy density $\kappa = \rho\tilde{\kappa}$, but what is the underlying process S, for which the pair (S,κ) is a tL model? Assuming that S satisfies (17), the only thing that is required for the consistency, is the compensator specification in (40). Let us now show how to construct a pure jump martingale with given characteristics.

5.2 Jump Measure Specification

Assume that we are given a Poisson random measure N (an integer valued random measure with deterministic compensator) with compensator $\rho(x)dxdt$, where ρ is defined in (36). Notice that this particular form of the compensator is not crucial for our derivations, as long as the compensator is absolutely continuous, takes finite values on the sets $(\mathbb{R}\setminus[-\varepsilon,\varepsilon])\times[0,t]$, and is equal to infinity on $([-\varepsilon,\varepsilon]\setminus\{0\})\times[0,t]$, for any $\varepsilon > 0$ and $t \in (0,\bar{T}]$. We choose to use $\rho(x)dxdt$ in order to simplify some of the notation.

We construct the measure M corresponding to the jumps of the logarithm of the underlying, as having the same times of jump as N, but with, possibly, different jump sizes. In other words, if $\{T_n, x_n\}$ denote the atoms of N, then we assume that the atoms of M are given by $\{T_n, W(T_n,x_n)\}$, for some predictable random function $W : \Omega\times[0,\bar{T}]\times(\mathbb{R}\setminus\{0\}) \hookrightarrow \mathbb{R}$ (see Definition 1.3 in Section II.1 of [23]), which we need to specify.

In order for $\rho(x)\tilde{\kappa}_t(t,x)dxdt$ to be a compensator of M, it is necessary and sufficient that the following is satisfied: for any nonnegative predictable function $f : \Omega \times [0,\bar{T}] \times (\mathbb{R} \setminus \{0\}) \hookrightarrow \mathbb{R}$, we have

$$\mathbb{E} \int_{\mathbb{R}\times[0,\bar{T}]} f(\omega,t,x)\tilde{\kappa}_t(t,x)\rho(x)dxdt = \mathbb{E} \int_{\mathbb{R}\times[0,\bar{T}]} f(\omega,t,x)\, M(dx,dt).$$

Notice that, by our assumption on the form of M, the above right hand side is equal to

$$\mathbb{E} \int_{\mathbb{R}\times[0,\bar{T}]} f(\omega,t,W(t,x))\, N(dx,dt),$$

which in turn, by the definition of a compensator (and because W is predictable), is equal to

$$\mathbb{E} \int_{\mathbb{R}\times[0,\bar{T}]} f(\omega,t,W(t,x))\, \rho(x)dxdt.$$

Thus, we need to find a predictable function W such that, for any nonnegative predictable f, we have

$$\mathbb{E} \int_{\mathbb{R}\times[0,\bar{T}]} f(\omega,t,x)\tilde{\kappa}_t(t,x)\rho(x)dxdt = \mathbb{E} \int_{\mathbb{R}\times[0,\bar{T}]} f(\omega,t,W(t,x))\, \rho(x)dxdt. \tag{42}$$

Such a function W may not be unique since the random measure M is not uniquely determined by its compensator. However, now with a possible loss of generality, we choose a specific form of W, which satisfies (42). First, we introduce functions

$$F_t(x) = \int_x^{\text{sign}(x)\infty} \tilde{\kappa}_t(t,y)\rho(y)dy, \quad G(x) = \int_x^{\text{sign}(x)\infty} \rho(y)dy,$$

and make a change of variables in (42) to obtain

$$\mathbb{E} \int_{\mathbb{R}\times[0,\bar{T}]} f\left(\omega,t,F_t^{-1}(x)\right) dxdt = \mathbb{E} \int_{\mathbb{R}\times[0,\bar{T}]} f\left(\omega,t,W\left(t,G^{-1}(x)\right)\right) dxdt,$$

where $F_t^{-1}(\,\cdot\,)$ and $G^{-1}(\,\cdot\,)$ are the (right continuous) generalized inverse functions. Thus, the specification $W(t,x) = W^{\tilde{\kappa}_t}(x)$ with

$$W^{\tilde{\kappa}_t}(x) := F_t^{-1}\left(G(x)\right), \quad W^{\tilde{\kappa}_t}(0) := 0, \tag{43}$$

fulfills (42). An important property of representation (43) is that $W^{\tilde{\kappa}_t}$ is expressed through $\tilde{\kappa}_t$ in a deterministic manner. In particular, it implies that $W^{\tilde{\kappa}_t}(x)$ is indeed predictable. Therefore, the integer valued random measure M, defined by its atoms $\left\{T_n, W^{\tilde{\kappa}_{T_n}}(x_n)\right\}$, has the compensator $\tilde{\kappa}_t(t,x)\rho(x)dxdt$. Notice also that by construction, $W^{\tilde{\kappa}_\cdot}(\,\cdot\,)$, as a random function, is locally integrable with respect to N (see II.1.27 in [23] for the definition of such an integrability).

5.3 Existence Result

Making use of the above constructions, we restrict our framework to dynamics of $(S_t, \tilde{\kappa}_t)_{t\in[0,\bar{T}]}$ of the form

$$\begin{cases} \tilde{\kappa}_t = \tilde{\kappa}_0 + \int_0^t Q^{\tilde{\beta}_u} \mathbf{1}_{u\le\tilde{\tau}_0} du + \sum_{n=1}^m \int_0^t \tilde{\beta}_u^n \mathbf{1}_{u\le\tilde{\tau}_0} dB_u^n, \\ S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_{u-} \left(\exp\left(W^{\tilde{\kappa}_t}(x)\right) - 1\right) \left(N(dx,du) - \rho(x)dxdu\right), \end{cases} \tag{44}$$

where ρ is defined in (36), $\tilde{\tau}_0$ is given by (37), $W^{\tilde{\kappa}_t}$ is defined in (43), $Q^{\tilde{\beta}_t}$ is given by (41), $B = \left(B^1, \ldots, B^m\right)$ is a multidimensional Brownian motion, N is a Poisson random measure (with compensator $\rho(x)dxdt$), each $\tilde{\beta}^n$ is a progressively measurable square integrable random process with values in $\tilde{\mathcal{H}}$, and the stochastic integrals in (44) are understood as their cádlág modifications.

Finally, we need to make sure that the martingale property of the underlying price S (which was imposed exogenously in Section 3) is satisfied. In general, S, given by (44), is a martingale if and only if the following holds

$$\mathbb{E} \int_0^{\bar{T}} \int_{\mathbb{R}} S_{u-} \left(\exp\left(W^{\tilde{\kappa}_t}(x)\right) - 1\right) \left(N(dx,du) - \rho(x)dxdu\right) = 0. \tag{45}$$

To see this, recall that S is a positive local martingale (see (18)), and repeat the argument presented in Subsection 2.1.

Notice that, if $\tilde{\kappa}$ is independent of N, the process $X_t = \log\left(S_t/S_0\right)$ has conditionally independent increments with respect to the σ-algebra generated by $(\tilde{\kappa}_t)_{t\in[0,\bar{T}]}$. Applying the Theorem II.6.6 in [23], we conclude that the conditional distribution of $X_{\bar{T}}$, given $(\tilde{\kappa}_t)_{t\in[0,\bar{T}]}$, is the one of the corresponding additive process at time $\bar{T}$. Then, using the argument presented in Section 2 (recall (6)), we conclude that the respective conditional expectation of $\exp\left(X_{\bar{T}}\right)$ is equal to one, which yields (45). Thus, in view of (44), the martingale property of S can be guaranteed by assuming that $\tilde{\beta}$ and the Brownian motion B are independent of the Poisson random measure N.

Remark 16 It may seem too restrictive to require that $\tilde{\kappa}$ is independent of the measure N, which governs the arrival of jumps. In fact, it could be interesting to consider models in which the behavior of the intensity changes, when large jumps occur. Then, in order to guarantee the martingale property, we can use the version of Novikov condition, given in Remark 5, which in the present setup rewrites as

$$\mathbb{E} \exp\left(\frac{e}{2} \int_0^{\bar{T}} \|\tilde{\kappa}_t(t,\,\cdot)\|_{\tilde{\mathcal{B}}^0} dt\right) < \infty.$$

Finally, we can formulate the desired existence result.

Theorem 17 *For any given Poisson random measure N, with compensator $\rho(x)dxdt$, any Brownian motion $B = \left(B^1, \ldots, B^m\right)$ independent of N, and any progressively measurable square integrable $\tilde{\mathcal{H}}$-valued stochastic processes $\left\{\tilde{\beta}^n\right\}_{n=1}^m$ independent of N, there exists a unique (up to indistinguishability) pair $(S_t, \tilde{\kappa}_t)_{t\in[0,\bar{T}]}$ of processes satisfying (44). The pair $(S_t, \rho\tilde{\kappa}_t)_{t\in[0,\bar{T}]}$ gives a tangent Lévy model.*

Proof:

The construction presented before Lemma 15 provides $\tilde{\kappa}$ satisfying the first line of (44). This construction is clearly unique given $\tilde{\beta}$ and B, and the resulting dynamic Lévy density $\kappa = \rho\tilde{\kappa}$ satisfies the drift restriction (34). Given $\tilde{\kappa}$ and N, the process S is uniquely defined by the second line of (44), and, by construction, it satisfies (17) and the compensator specification (26). Moreover, by the argument presented before Remark 16, under the independence assumption of the theorem, the process S is a martingale. A simple application of Corollary 13 completes the proof. ■

Notice that in some sense, the above theorem provides a local existence result: (44) implies that the process $\tilde{\kappa}$ stops at $\tilde{\tau}_0$, and from this time on, the underlying evolves as the exponential of a process with independent increments. Notice that this does not necessarily lead to any pathological behavior of the underlying since most likely, $\tilde{\kappa}_{\tilde{\tau}_0}(T,x)$ is equal to zero at only "few" points (T,x), so that the resulting Lévy density is not degenerate. However, the need to stop $\tilde{\kappa}$ at $\tilde{\tau}_0$ may not be a desirable property, in particular if one is looking for some kind of stationarity in the model. Therefore, it is reasonable to consider the diffusion coefficients $\left\{\tilde{\beta}_t^n\right\}_{n=1}^m$ (and therefore $\tilde{\alpha}_t$) as functions of $\tilde{\kappa}_t$, so that the resulting dynamics of $\tilde{\kappa}$ guarantee that it always stays positive (in other words, $\tilde{\tau}_0 = \infty$ almost surely). In such case, it is also possible to make $\tilde{\beta}_t$, and therefore $\tilde{\kappa}_t$, depend upon S_t. Then, of course, the independence assumption of Theorem 17 would be violated, and we would need to make sure that for example, the dynamics of $\tilde{\kappa}_t$ are such that $\|\tilde{\kappa}_t\|_{\tilde{\mathcal{B}}}$ is bounded over $t \in [0,\bar{T}]$ by a constant in order to use Remark 16. In addition, the system (44) would become a "true" system of equations for S and $\tilde{\kappa}$ (when all the terms in the right hand side have a nontrivial dependence upon the left hand side, unlike it is in the present setup), and the questions of existence and uniqueness of the solution would be significantly more complicated. In the present paper, we do not provide the analysis of this problem in full generality. However, Section 6 illustrates the above discussion with an example of a tL model $(S_t, \rho\tilde{\kappa}_t)_{t\in[0,\bar{T}]}$, in which $\tilde{\kappa}$ is constructed to stay positive at all times.

6 Example of a Tangent Lévy Model and Implementation

In this section, we give an explicit example of a tangent Lévy model which does not need to be stopped before $\bar{T}$. We pick $\lambda > 1$, $\lambda' > 0$, $\delta \in (0,1)$ and assume that we are in the setup of Section 5, in particular, the dynamics of the model are given by (44). Then, according to Theorem 17, in order to construct a tL model, we only need to specify the progressively measurable and square integrable processes $\left\{\tilde{\beta}^n\right\}_{n=1}^m$ with values in $\tilde{\mathcal{H}}$.

We choose $m = 1$ and use the notation $\tilde{\beta}$ for $\tilde{\beta}^1$, which is specified in the following way

$$\tilde{\beta}_t(T,x) = \gamma_t C(x),$$

where

$$C(x) = \operatorname{sign}(x) e^{-\lambda'|x|} \left(|x|^{1+2\delta} \wedge 1\right) \left(\left(\lambda + \lambda'\right)|x|^{1-\delta} - (1-\delta)|x|^{-\delta}\right), \qquad (46)$$

and γ is some scalar random process which will be specified later. This particular function C is only chosen for its mathematical convenience: the integral of ρC can be computed in closed form, and more importantly, ρC is locally integrable at zero, which will allow for further simplification of the drift restriction. But as it will become clear later on, the algorithm described below works for any other function from $\tilde{\mathcal{H}}^0$ (see the definition in Section 5). Notice that with the above specification, we have

$$\beta_t(T,x) = \gamma_t \rho(x) C(x),$$
$$\bar{\tilde{\beta}}_t(T,x) = \gamma_t (T - t \wedge T) C(x),$$

where ρ is defined in (36).

Now we compute $\tilde{\alpha}$ from the drift restriction. Since function ρC is absolutely integrable, $\partial^2_{x^2}\psi\left(\rho\bar{\tilde{\beta}}^n_t(T);x\right)$ and $\partial^2_{x^2}\psi\left(\rho\tilde{\beta}^n_t(T);x\right)$ are absolutely integrable on $\mathbb{R}$ as functions of x. Then, integrating by parts in (41), one obtains

$$Q^{\tilde{\beta}_t}(T,x) = -\frac{e^{-x}}{\rho(x)} \int_{\mathbb{R}} \partial^2_{y^2}\psi\left(\rho\bar{\tilde{\beta}}_t(T);y\right)\partial^2_{x^2}\psi\left(\rho\tilde{\beta}_t(T);x-y\right) - \partial_y\psi\left(\rho\bar{\tilde{\beta}}_t(T);y\right)\partial_x\psi\left(\rho\tilde{\beta}_t(T);x-y\right)dy, \tag{47}$$

which provides the simplest form of the drift restriction (recall (40)). Let's compute the following auxiliary components:

$$\partial_x\psi\left(\rho C;x\right) = e^x h(x), \quad \partial^2_{x^2}\psi\left(\rho C;x\right) = e^x\left(h(x)+f(x)\right),$$

in the notation

$$f(x) = \operatorname{sign}(x) e^{-(\lambda+\lambda')|x|}\left(\left(\lambda+\lambda'\right)|x|^{1-\delta} - (1-\delta)|x|^{-\delta}\right),$$
$$h(x) = -|x|^{1-\delta} e^{-(\lambda+\lambda')|x|}.$$

Now we recall the form of $\tilde{\beta}$ and $\bar{\tilde{\beta}}$, and, plugging the above expressions into (47), obtain

$$Q^{\tilde{\beta}_t}(T,x) = \gamma_t^2\left(T - t\wedge T\right)A(x),$$

where

$$A(x) = -e^{\lambda|x|}\left(|x|^{1+2\delta}\wedge 1\right)\int_{\mathbb{R}}\left(f(y)+2h(y)\right)f(x-y)dy \tag{48}$$

As announced, we construct $(\tilde{\kappa}_t)_{t\in[0,\bar{T}]}$, so that it stays nonnegative (even positive) at all times. In order to preserve nonnegativity, we let γ_t depend upon $\tilde{\kappa}_t$, namely, we choose the following specification

$$\gamma_t = \gamma(\tilde{\kappa}_t, t) := \frac{\sigma}{\epsilon}\left(\inf_{T\in[t,\bar{T}],x\in\mathbb{R}}\tilde{\kappa}_t(T,x)\wedge\epsilon\right), \tag{49}$$

where σ and ϵ are some positive constants. Then the process $\tilde{\kappa}$ is defined as the unique strong solution of the following infinite dimensional stochastic differential equation

$$d\tilde{\kappa}_t(T,x) = \gamma^2(\tilde{\kappa}_t,t)\left(T-t\wedge T\right)A(x)dt + \gamma(\tilde{\kappa}_t,t)C(x)dB_t, \tag{50}$$

where A, C and γ are given in (48), (46) and (49) respectively. The solution is well defined since function $\gamma : \tilde{\mathcal{B}}\times[0,\bar{T}]\hookrightarrow\mathbb{R}$ is globally Lipschitz in the first variable, uniformly over the second one, and bounded (see, for example, Theorem 7.4 in [14]). Then the following proposition shows that, almost surely

$$\forall t\in[0,\bar{T}], \quad \inf_{x\in\mathbb{R},T\in[t,\bar{T}]}\tilde{\kappa}_t(T,x)\geq 0. \tag{51}$$

Proposition 18 *With positive initial condition, the process $\tilde{\kappa}$, defined by (50) is almost surely nonnegative in the sense of (51).*

Proof:
Since $\tilde{\kappa}_t$ takes values in the space of continuous functions, it is enough to show nonnegativity of $(\tilde{\kappa}_t(T,x))_{t\in[0,T]}$ for any $x \in \mathbb{R}$ and $T \in (0,\bar{T}]$.

Notice that the process γ is continuous. Then the stopping times $\tau_n := \inf\{t : \gamma_t \leq 1/n\}$ are well defined for any integer $n \geq 1$. The process $\tilde{\kappa}_\cdot(T,x)$, stopped at τ_n, is strictly positive, therefore, its logarithm is correctly defined. Using Ito's formula, we obtain

$$d\left[\log \tilde{\kappa}_{t\wedge\tau_n}(T,x)\right] = X_t^n dt + Y_t^n dB_t,$$

where

$$X_t^n = \left(\frac{\gamma^2(\tilde{\kappa}_t,t)\,(T - t\wedge T)\,A(x)}{\tilde{\kappa}_t(T,x)} - \frac{\gamma^2(\tilde{\kappa}_t,t)C^2(x)}{2\tilde{\kappa}_t^2(T,x)}\right)\mathbf{1}_{t\leq\tau_n},$$
$$Y_t^n = \frac{\gamma(\tilde{\kappa}_t,t)C(x)}{\tilde{\kappa}_t(T,x)}\mathbf{1}_{t\leq\tau_n}.$$

Notice that the ratios above are well defined, since, almost surely, $\tilde{\kappa}_t(T,x)$ is positive for $t\in[0,\tau_n]$. Let us now show that we have, almost surely

$$\sup_{n\geq1}|\log \tilde{\kappa}_{T\wedge\tau_n}| < \infty. \tag{52}$$

To see this, first notice that

$$\left|\int_0^T X_t^n dt\right| \leq \frac{\sigma^2}{\epsilon}T^2 A(x) + \frac{\sigma^2}{\epsilon^2}TC^2(x), \tag{53}$$

almost surely. Then, for each $n \geq 1$, consider the martingale M^n, given by

$$M_t^n = \int_0^t Y_t^n dB_t.$$

These are true martingales, since

$$\int_0^T \left(Y_t^n\right)^2 dt \leq \frac{\sigma^2}{\epsilon^2}TC^2(x),$$

almost surely. Moreover using Doob's maximal inequality, we obtain

$$\mathbb{E}\left(\sup_{t\in[0,T]} M_t^n\right)^2 \leq 4\mathbb{E}\left(M_T^n\right)^2 \leq 4\frac{\sigma^2}{\epsilon^2}TC^2(x).$$

Denoting

$$M^* := \lim_{n\to\infty}\sup_{t\in[0,T]} M_t^n,$$

which is well defined for almost all ω, since τ_n is almost surely nondecreasing, and the dentity

$$M_t^{n+1}\mathbf{1}_{t\leq\tau_n} = M_t^n\mathbf{1}_{t\leq\tau_n}$$

implies that $\sup_{t\in[0,T]} M_t^n$ is almost surely nondecreasing in n, the monotone convergence theorem yields

$$\mathbb{E}\left(M^*\right)^2 = \lim_{n\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} M_t^n\right)^2 < \infty,$$

which implies that M^* is finite almost surely, and the latter, together with (53), yields (52).

It only remains to notice that, if $\lim_{n\to\infty} \tau_n \leq T$, then $\sup_{n\geq 1} |\log \tilde{\kappa}_{T\wedge\tau_n}| = \infty$. Therefore, almost surely, there exists n, such that $\tau_n > T$, which implies that $\tilde{\kappa}_t(T,x)$ is almost surely nonnegative (even positive) for all $t \in [0,T]$. ∎

Defined by (50), the process $\tilde{\kappa}$ satisfies the first line of (44), and $\tilde{\tau}_0 = \infty$ almost surely. Therefore, choosing a Poisson random measure N (with compensator ρ), independent of the Brownian motion B which drives the dynamics of $\tilde{\kappa}$, we define the underlying S via the second line of (44) to obtain $(S_t, \rho\tilde{\kappa}_t)_{t\in[0,\bar{T}]}$, and, applying Theorem 17 we conclude that we have the desired example of a tL model.

The above example demonstrates the machinery that can be used to construct tL models, with $\tilde{\beta}_t^n(T,x)$ being proportional to some fixed deterministic function $C(x)$. In fact, this construction can be generalized to functions of the form $C(T,x)$. Notice nevertheless that the particular form of C we chose in this example implies that the Brownian motion B moves the intensities of positive and negative jumps of the underlying in opposite directions. In general, it seems reasonable to combine $\tilde{\beta}^n(\cdot,\cdot)$'s given by functions "$C$" of different shapes. These functions, $\{C^n\}$, would correspond to different Brownian motions and may have different stochastic factors $\{\gamma^n\}$. An important question is then the choice of the appropriate functions C^n. We do not elaborate on this important practical problem in the present paper. However, we suggest that the functions C^n can be obtained from the analysis in principal components (PCA) of the time series of $\tilde{\kappa}_t(\cdot,\cdot)$, fitted to the historical call prices on dates t of a recent past, Notice that assuming that C^n's are deterministic implies that they don't change as we revert back from $\mathbb{Q}$ to the real-world measure.

7 Conclusion and Future Research

In this paper, we introduced a new class of market models based on European call options. Consistent with the market model philosophy, these models allow to start with the observed surface of call prices and prescribe explicitly its future stochastic dynamics under the risk-neutral measure. In particular, such dynamics do not produce arbitrage, and for example, can be used to simulate the future (arbitrage-free) evolution of the implied volatility surface in a rather flexible way. This is in stark contrast with the classical models in which the implied volatility surface has very rigid dynamics. We outlined the main steps of a possible implementation algorithm, and provided a specific example.

Unlike the models of dynamic local volatility considered in [3] and [2], the present framework is consistent with the assumption that the underlying is given by a pure jump process. Therefore, the classes of *tangent Lévy* and *dynamic local volatility* models do not intersect, except for some degenerate cases.

Although it is clear that by definition, a tangent Lévy model implies that the underlying is a pure jump martingale, one naturally would like to describe explicitly the set of all possible underlying dynamics that can be generated by tangent Lévy models. Addressing this issue, the first and somehow simpler question is: *what are the possible underlying risk-neutral dynamics which produce call price surfaces that can be represented through some* *time-inhomogeneous Lévy density*? In other words, we would like to characterize the class of stochastic processes whose marginal distributions can be mimicked by some exponential additive process. As discussed in the introduction, the answer to analogous question in the continuous case was provided by Gyöngy [19], whose results imply that, under some technical conditions, call price surfaces produced by underlying Itô processes can be represented via a local volatility code-book. Unfortunately, there is very little hope that by imposing some technical assumptions, we can guarantee that every pure jump martingale has marginal distributions of some exponential additive process, since in particular, this would imply that these marginal distributions are *infinitely divisible* (see [8] for an alternative representation of the one-dimensional distributions of semimartingales with jumps). Nevertheless, for practical purposes, considering only infinitely divisible distributions is of course sufficient since the full marginal distributions of the underlying are never known precisely.

Finally, we would like to mention a possible extension which would allow the resulting models to have some qualitatively different characteristics, and as we believe, can be obtained by following the program outlined in the present paper. Namely, we suggest that instead of considering the code-book consisting of the Lévy density alone, one could also include a constant, which would have the meaning of the ”instantaneous volatility”. In this case, the marginal distributions of the logarithm of the underlying would be reproduced by an additive process with a nontrivial Brownian motion component, and it would make it possible to allow the underlying to have a nonzero continuous martingale part. The extended code-book, consisting of the Lévy density and the (scalar) ”volatility”, can then be put in motion, and one can try to derive the corresponding consistency conditions using the techniques presented in this paper.

8 Appendix A

Fix some $T > 0$ and $t \in [0, T)$. Denote

$$\bar{\kappa}(x) = \int_t^T \frac{\kappa(u, x)}{T - t} du.$$

Due to (4), we can apply Fubini's theorem and obtain

$$\int_{T \wedge t}^T \hat{\psi}(\kappa(u); x) du = (T - t) \int_{\mathbb{R}} e^{-2\pi i x y} \psi(\bar{\kappa}; y) dy.$$

Now, using integration by parts twice, we can simplify the integral in the right hand side of the above. First, over the positive half line

$$\begin{aligned}\int_0^\infty e^{-2\pi i x y} \psi(\bar{\kappa}; y) dy &= \int_0^\infty \partial_y \left(\frac{e^{-2\pi i x y}}{-2\pi i x} \right) \int_y^\infty \left(e^z - e^y \right) \bar{\kappa}(z) dz dy \\ &= -\frac{1}{2\pi i x} \int_0^\infty \left(e^{y(1-2\pi i x)} - e^y \right) \int_y^\infty \bar{\kappa}(z) dz dy \\ &= -\frac{1}{2\pi i x(1 - 2\pi i x)} \int_0^\infty \left(e^{y(1-2\pi i x)} - e^y (1 - 2\pi i x) - 2\pi i x \right) \bar{\kappa}(y) dy.\end{aligned}$$

And similarly proceed with the negative half line. As a result, we obtain

$$\int_{\mathbb{R}} e^{-2\pi ixy}\psi(\bar{\kappa};y)dy$$
$$= -\frac{1}{2\pi ix(1-2\pi ix)}\int_{\mathbb{R}}\left(e^{y(1-2\pi ix)} - e^{y}(1-2\pi ix) - 2\pi ix\right)\bar{\kappa}(y)dy.$$

On the other hand, according to the Lévy-Khinchine formula

$$\mathbb{E}\left(e^{i(-i-2\pi x)\log\tilde{S}_T}\middle|\log\tilde{S}_t = 0\right)$$
$$= \exp\left[(T-t)\int_{\mathbb{R}}\left(e^{y(1-2\pi ix)} - e^{y}(1-2\pi ix) - 2\pi ix\right)\bar{\kappa}(y)dy\right],$$

which yields (14).

9 Appendix B

Proof of Lemma 15*:*

Throughout this proof, $\tilde{\alpha}_t := Q^{\tilde{\beta}_t}$. We need to show that $\tilde{\alpha}_t(\,\cdot\,,\,\cdot\,) \in \tilde{\mathcal{B}}$ and its $\tilde{\mathcal{B}}$-norm is integrable in $t \in [0,\bar{T}]$. The fact that $\tilde{\alpha}$ is progressively measurable follows from its representation through $\tilde{\beta}$.

Let's rewrite (41) in the following form

$$\tilde{\alpha}_t(T,x) = -e^{\lambda|x|-x}\left(|x|\wedge 1\right)^{1+2\delta}\sum_{n=1}^{N}\int_{\mathbb{R}}\left\{\partial^3_{y^3}\psi\left(w(\,\cdot\,)\bar{\hat{\beta}}^n_t(T,\,\cdot\,);y\right)\cdot\right.$$
$$\left[\partial_x\psi\left(w(\,\cdot\,)\hat{\beta}^n_t(T,\,\cdot\,);x-y\right) - (1-y\partial_x)\,\partial_x\psi\left(w(\,\cdot\,)\hat{\beta}^n_t(T,\,\cdot\,);x\right)\right]$$
$$\left. - \partial_y\psi\left(w(\,\cdot\,)\bar{\hat{\beta}}^n_t(T,\,\cdot\,);y\right)\partial_x\psi\left(w(\,\cdot\,)\hat{\beta}^n_t(T,\,\cdot\,);x-y\right)\right\}dy, \tag{54}$$

where

$$w(x) = e^{-(\lambda+\lambda')|x|}\left(|x|\wedge 1\right)^{-1-\delta},$$
$$\hat{\beta}^n_t(T,x) = e^{\lambda'|x|}\left(|x|\wedge 1\right)^{-\delta}\tilde{\beta}^n_t(T,x),$$
$$\bar{\hat{\beta}}^n_t(T,x) = e^{\lambda'|x|}\left(|x|\wedge 1\right)^{-\delta}\int_{t\wedge T}^{T}\tilde{\beta}^n_t(u,x)du.$$

Notice that $\hat{\beta}^n_t(T,\,\cdot\,)$ and $\bar{\hat{\beta}}^n_t(T,\,\cdot\,)$ are in $\mathcal{G}$, and their $\mathcal{G}$-norms are estimated by the $\tilde{\mathcal{H}}^0$-norms of $\tilde{\beta}^n_t(T,\,\cdot\,)$ and $\bar{\tilde{\beta}}^n_t(T,\,\cdot\,)$ respectively.

We will need the following auxiliary estimates

$$\left|\partial_x\psi\left(w(\,\cdot\,)\hat{\beta}^n_t(T,\,\cdot\,);x\right)\right| \le e^x\int_{|x|}^{\infty} w(z)\left|\hat{\beta}^n_t(T,\mathrm{sign}(x)z)\right|dz$$
$$\le c_3\|\tilde{\beta}^n_t(T,\,\cdot\,)\|_{\tilde{\mathcal{H}}^0}e^{x-(\lambda+\lambda')|x|}\left(|x|\wedge 1\right)^{-\delta}, \tag{55}$$
$$\left|\partial^k_{x^k}\psi\left(w(\,\cdot\,)\hat{\beta}^n_t(T,\,\cdot\,);x\right)\right| \le c_4\|\tilde{\beta}^n_t(T,\,\cdot\,)\|_{\tilde{\mathcal{H}}^0}e^{x-(\lambda+\lambda')|x|}\left(|x|\wedge 1\right)^{1-k-\delta},\quad k=2,3,$$

which also hold for $\bar{\hat{\beta}}^n_t$, with $\tilde{\mathcal{H}}$-norm instead of $\tilde{\mathcal{H}}^0$-norm.

Let us now estimate the terms inside the sum in the right hand side of (54). For now, we fix some $t \in [0, \bar{T})$, $T \in (t, \bar{T}]$ and $n \in \{1, \dots, m\}$. The corresponding term in (54) has a form of an integral, let's concentrate on the first part of the integrand. Namely, we denote

$$I^1(T,x) = \int_{\mathbb{R}} \partial^3_{y^3}\psi\left(w(\,\cdot\,)\bar{\tilde{\beta}}^n_t(T,\,\cdot\,);y\right)\cdot \\ \left[\partial_x\psi\left(w(\,\cdot\,)\hat{\beta}^n_t(T,\,\cdot\,);x-y\right) - (1-y\partial_x)\,\partial_x\psi\left(w(\,\cdot\,)\hat{\beta}^n_t(T,\,\cdot\,);x\right)\right]dy. \tag{56}$$

For notational convenience, we introduce $\tilde{\lambda} := \lambda + \lambda' > 1$. We now split the domain of integration into the following parts:

$$I^{1,1}(T,x) = \mathrm{sign}(x)\int_{-\mathrm{sign}(x)\infty}^{-\frac{1}{4}\mathrm{sign}(x)(|x|\wedge 1)}(*)dy,\;\; I^{1,2}(T,x) = \int_{-\frac{1}{4}(|x|\wedge 1)}^{\frac{1}{4}(|x|\wedge 1)}(*)dy,$$
$$I^{1,3}(T,x) = \mathrm{sign}(x)\int_{\frac{1}{4}\mathrm{sign}(x)(|x|\wedge 1)}^{x-\frac{1}{4}\mathrm{sign}(x)(|x|\wedge 1)}(*)dy, \tag{57}$$
$$I^{1,4}(T,x) = \int_{x-\frac{1}{4}(|x|\wedge 1)}^{x+\frac{1}{4}(|x|\wedge 1)}(*)dy,\;\; I^{1,5}(T,x) = \mathrm{sign}(x)\int_{x+\frac{1}{4}\mathrm{sign}(x)(|x|\wedge 1)}^{\mathrm{sign}(x)\infty}(*)dy,$$

where $(*)$ is the integrand in the right hand side of (56). Let's estimate $I^{1,5}$, making use of (55)

$$\left|I^{1,5}(T,x)\right| \le c_5\|\tilde{\beta}^n_t\|^2_{\tilde{\mathcal{H}}}\mathrm{sign}(x)\int_{x+\frac{1}{4}\mathrm{sign}(x)(|x|\wedge 1)}^{\mathrm{sign}(x)\infty} e^{y-\tilde{\lambda}|y|}\left(|y|\wedge 1\right)^{-2-\delta}\cdot \\ \left[e^{x-y-\tilde{\lambda}|x-y|}\left(|x-y|\wedge 1\right)^{-\delta} + e^{x-\tilde{\lambda}|x|}\left(|x|\wedge 1\right)^{-\delta}\left(1+|y|\left(|x|\wedge 1\right)^{-1}\right)\right]dy \\ \le c_6\|\tilde{\beta}^n_t\|^2_{\tilde{\mathcal{H}}}\left(e^x\left(|x|\wedge 1\right)^{-\delta}\int_{|x|+\frac{1}{4}(|x|\wedge 1)}^{\infty} e^{-\tilde{\lambda}|y|}\left(|y|\wedge 1\right)^{-2-\delta}dy \right. \\ \left. + e^{x-\tilde{\lambda}|x|}\sum_{k=0}^{1}\left(|x|\wedge 1\right)^{-k-\delta}\int_{|x|+\frac{1}{4}(|x|\wedge 1)}^{\infty} e^{-(\tilde{\lambda}-1)|y|}\left(|y|\vee 1\right)^k\left(|y|\wedge 1\right)^{k-2-\delta}dy\right) \\ \le c_7\|\tilde{\beta}^n_t\|^2_{\tilde{\mathcal{H}}}e^{x-\tilde{\lambda}|x|}\left(|x|\wedge 1\right)^{-1-2\delta}$$

Similarly, we proceed with the first integral

$$\left|I^{1,1}(T,x)\right| \le c_8\|\tilde{\beta}^n_t\|^2_{\tilde{\mathcal{H}}}\mathrm{sign}(x)\int_{-\mathrm{sign}(x)\infty}^{-\frac{1}{4}\mathrm{sign}(x)(|x|\wedge 1)} e^{y-\tilde{\lambda}|y|}\left(|y|\wedge 1\right)^{-2-\delta}\cdot \\ \left[e^{x-y-\tilde{\lambda}|x-y|}\left(|x-y|\wedge 1\right)^{-\delta} + e^{x-\tilde{\lambda}|x|}\left(|x|\wedge 1\right)^{-\delta}\left(1+|y|\left(|x|\wedge 1\right)^{-1}\right)\right]dy \\ \le c_9\|\tilde{\beta}^n_t\|^2_{\tilde{\mathcal{H}}}e^{x-\tilde{\lambda}|x|}\left(|x|\wedge 1\right)^{-1-2\delta}$$

In the same way we can estimate the third integral

$$\left|I^{1,3}(T,x)\right| \le c_{10}\|\tilde\beta_t^n\|_{\tilde{\mathcal H}}^2 \operatorname{sign}(x)\int_{\frac14\operatorname{sign}(x)(|x|\wedge1)}^{x-\frac14\operatorname{sign}(x)(|x|\wedge1)} e^{y-\tilde\lambda|y|}\left(|y|\wedge1\right)^{-2-\delta}\cdot$$
$$\left[e^{x-y-\tilde\lambda|x-y|}\left(|x-y|\wedge1\right)^{-\delta}+e^{x-\tilde\lambda|x|}\left(|x|\wedge1\right)^{-\delta}\left(1+|y|\left(|x|\wedge1\right)^{-1}\right)\right]dy$$
$$\le c_{11}\|\tilde\beta_t^n\|_{\tilde{\mathcal H}}^2 e^{x-\tilde\lambda|x|}\left(|x|\vee1\right)\left(|x|\wedge1\right)^{-1-2\delta}$$

Before providing estimates for the two remaining integrals, notice that, since $\hat\beta_t^n(T,x)$ is absolutely continuous function of x outside any neighborhood of zero, the same is true for

$$\partial_{x^k}^k\psi\left(w(\,\cdot\,)\hat\beta_t^n(T,\,\cdot\,);x\right),$$

with $k=1,2$. Then for $y\ne x\ne0$ we have

$$\left|\partial_x\psi\left(w(\,\cdot\,)\hat\beta_t^n(T,\,\cdot\,);x-y\right)-(1-y\partial_x)\,\partial_x\psi\left(w(\,\cdot\,)\hat\beta_t^n(T,\,\cdot\,);x\right)\right|$$
$$\le y^2\sup_{z\in[(x-y)\wedge x,(x-y)\vee x]}\left|\partial_{z^3}^3\psi\left(w(\,\cdot\,)\hat\beta_t^n(T,\,\cdot\,);z\right)\right|$$
$$\le c_{12}y^2\|\tilde\beta_t^n(T,\,\cdot\,)\|_{\tilde{\mathcal H}^0}\sup_{z\in[(x-y)\wedge x,(x-y)\vee x]}e^{z-\tilde\lambda|z|}\left(|z|\wedge1\right)^{-2-\delta}.$$

Thus, we continue

$$\left|I^{1,2}(T,x)\right|\le$$
$$c_{13}\|\tilde\beta_t^n\|_{\tilde{\mathcal H}}^2\int_{-\frac14(|x|\wedge1)}^{\frac14(|x|\wedge1)}e^{y-\tilde\lambda|y|}|y|^{-\delta}dy\sup_{z\in\left[x-\frac{(|x|\wedge1)}{4},x+\frac{(|x|\wedge1)}{4}\right]}\left(e^{z-\tilde\lambda|z|}\left(|z|\wedge1\right)^{-2-\delta}\right)$$
$$\le c_{14}\|\tilde\beta_t^n\|_{\tilde{\mathcal H}}^2\,e^{x-\tilde\lambda|x|}\left(|x|\wedge1\right)^{-1-2\delta}$$

And, finally

$$\left|I^{1,4}(T,x)\right|\le c_{15}\|\tilde\beta_t^n\|_{\tilde{\mathcal H}}^2\int_{x-\frac14(|x|\wedge1)}^{x+\frac14(|x|\wedge1)}e^{y-\tilde\lambda|y|}\left(|y|\wedge1\right)^{-2-\delta}\cdot$$
$$\left[e^{x-y-\tilde\lambda|x-y|}\left(|x-y|\wedge1\right)^{-\delta}+e^{x-\tilde\lambda|x|}\left(|x|\wedge1\right)^{-\delta}\left(1+|y|\left(|x|\wedge1\right)^{-1}\right)\right]dy$$
$$\le c_{16}\|\tilde\beta_t^n\|_{\tilde{\mathcal H}}^2e^{x-\tilde\lambda|x|}\left(|x|\wedge1\right)^{-2-\delta}\cdot$$
$$\int_{x-\frac14(|x|\wedge1)}^{x+\frac14(|x|\wedge1)}\left(|x-y|\wedge1\right)^{-\delta}+\left(|x|\wedge1\right)^{-\delta}+|y|\left(|x|\wedge1\right)^{-1-\delta}dy$$
$$\le c_{17}\|\tilde\beta_t^n\|_{\tilde{\mathcal H}}^2\,e^{x-\tilde\lambda|x|}\left(|x|\vee1\right)\left(|x|\wedge1\right)^{-1-2\delta}$$

The above estimates yield

$$\left|I^1(T,x)\right|\le c_{18}\|\tilde\beta_t^n\|_{\tilde{\mathcal H}}^2\,e^{x-\tilde\lambda|x|}\left(|x|\vee1\right)\left(|x|\wedge1\right)^{-1-2\delta},$$

It is also easy to see that the second term inside the integral in the right hand side of (54) is estimated in the same way. As a result, we obtain

$$|\tilde{\alpha}_t(T,x)| \le c_{19} \|\tilde{\beta}_t^n\|_{\tilde{\mathcal{H}}}^2 \, e^{-\lambda'|x|} \, (|x| \vee 1) \,, \tag{58}$$

which provides an upper bound on the $\mathcal{B}^0$-norm of $\tilde{\alpha}_t(T,\,\cdot\,)$.

Let's now show that $\tilde{\alpha}_t(T,\,\cdot\,)$ is continuous. To prove the continuity at zero, we need to show that the limit at $x=0$ exists. We will need the following useful relations, holding for all absolutely continuous functions f, with $\|f'\|_{\mathbb{L}^\infty(\mathbb{R})} < \infty$

$$\begin{aligned}
&\partial_x \psi\left(wf;x\right) = \operatorname{sign}(x)|x|^{-\delta}\frac{f(0)}{\delta}\mathbf{1}_{[0,1]}(|x|) + \underline{\underline{O}}\left(e^{-(\tilde{\lambda}-1)|x|}\left(|x|\wedge 1\right)^{1-\delta}\right),\\
&\partial^2_{x^2} \psi\left(wf;x\right) = |x|^{-1-\delta} f(0)\mathbf{1}_{[0,1]}(|x|) + \underline{\underline{O}}\left(e^{-(\tilde{\lambda}-1)|x|}\left(|x|\wedge 1\right)^{-\delta}\right),\\
&\partial^3_{x^3} \psi\left(wf;x\right) = -\operatorname{sign}(x)|x|^{-2-\delta}(1+\delta) f(0)\mathbf{1}_{[0,1]}(|x|) + \underline{\underline{O}}\left(e^{-(\tilde{\lambda}-1)|x|}\left(|x|\wedge 1\right)^{-1-\delta}\right),
\end{aligned} \tag{59}$$

where the first two equalities hold for all $x \in \mathbb{R}\setminus\{0\}$ and the last one is understood for almost every $x \in \mathbb{R}\setminus\{0\}$.

Now we are ready to proceed with the proof of the continuity at zero. As before, it is enough to consider I^1, defined by (56), the other term is treated similarly. Assume that $x \to 0$. In order to make use of (59), we need to split the domain of integration in I^1 into two parts: $[-|x|/2, |x|/2]$ and $\mathbb{R}\setminus[-|x|/2,|x|/2]$. For the integral over the second domain, we can apply (59) directly, but in the case of the integral around zero, we need to use integration by parts first:

$$\begin{aligned}
\int_{-\frac{|x|}{2}}^{\frac{|x|}{2}} (*) = &\left(\partial^2_{y^2}\psi\left(w(\,\cdot\,)\bar{\hat{\beta}}_t^n(T,\,\cdot\,);y\right)\cdot\right.\\
&\left.\left[\partial_x\psi\left(w(\,\cdot\,)\hat{\beta}_t^n(T,\,\cdot\,);x-y\right) - (1-y\partial_x)\,\partial_x\psi\left(w(\,\cdot\,)\hat{\beta}_t^n(T,\,\cdot\,);x\right)\right]\right)\Big|_{y=-\frac{|x|}{2}}^{y=\frac{|x|}{2}} +\\
&\int_{-\frac{|x|}{2}}^{\frac{|x|}{2}} \partial^2_{y^2}\psi\left(w(\,\cdot\,)\bar{\hat{\beta}}_t^n(T,\,\cdot\,);y\right)\partial^2_{x^2}\left[\psi\left(w(\,\cdot\,)\hat{\beta}_t^n(T,\,\cdot\,);x-y\right) - \psi\left(w(\,\cdot\,)\hat{\beta}_t^n(T,\,\cdot\,);x\right)\right] dy
\end{aligned}$$

After integrating by parts once more, we can apply (59). As a result, we obtain, as $x \to 0$

$$\begin{aligned}
&I^1(T,x) \qquad\qquad \text{(60)}\\
&= \frac{1+\delta}{\delta}\bar{\hat{\beta}}_t^n(T,0)\hat{\beta}_t^n(T,0)\int_{-1}^{1} -\operatorname{sign}(y)|y|^{-2-\delta}\left(\operatorname{sign}(x-y)|x-y|^{-\delta}\mathbf{1}_{[0,1]}(|x-y|)\right.\\
&\qquad\left. -\operatorname{sign}(x)|x|^{-\delta} - \delta y|x|^{-1-\delta}\right) dy + \underline{\underline{O}}\left(|x|^{-2\delta}\right) \qquad\qquad \text{(61)}\\
&= \frac{1+\delta}{\delta}\bar{\hat{\beta}}_t^n(T,0)\hat{\beta}_t^n(T,0)|x|^{-1-2\delta}\int_{-\frac{1}{|x|}}^{\frac{1}{|x|}} \operatorname{sign}(y)|y|^{-2-\delta}\cdot\\
&\qquad\left(1+\delta y - \operatorname{sign}(1-y)|1-y|^{-\delta}\mathbf{1}_{[0,\frac{1}{|x|}]}(|1-y|)\right) dy + \underline{\underline{O}}\left(|x|^{-2\delta}\right)\\
&= \frac{1+\delta}{\delta}\bar{\hat{\beta}}_t^n(T,0)\hat{\beta}_t^n(T,0)|x|^{-1-2\delta}\left(1+\bar{\bar{o}}(1)\right)\cdot\\
&\qquad\int_{\mathbb{R}} \operatorname{sign}(y)|y|^{-2-\delta}\left(1+\delta y - \operatorname{sign}(1-y)|1-y|^{-\delta}\right) dy,
\end{aligned}$$

where the last equality was obtained by splitting the domain of integration and applying the dominated convergence theorem.

Continuity of $I^1(T,\cdot)$ at any other point follows from the dominated convergence theorem. Thus, we conclude that $\tilde{\alpha}_t(T,\cdot)$ is continuous.

Now, applying Fubini's theorem, we can compute the partial T-derivative of $\tilde{\alpha}_t(\cdot,\cdot)$, say $\partial_T\tilde{\alpha}_t(T,x)$, defined pointwise at each x, for almost every $T\in(0,\bar{T})$. Then the continuity of $\partial_T\tilde{\alpha}_t(T,\cdot)$ can be shown in the same way as for $\tilde{\alpha}_t(T,\cdot)$ above. Moreover, repeating, essentially, the derivation of (58), we obtain

$$|\partial_T\tilde{\alpha}_t(T,x)|\leq c_{19}\left(\|\tilde{\beta}^n_t\|^2_{\tilde{\mathcal{H}}}+\|\tilde{\beta}^n_t\|_{\tilde{\mathcal{H}}}\left\|\frac{d}{dT}\tilde{\beta}^n_t(T)\right\|_{\tilde{\mathcal{H}}^0}\right)e^{-\lambda'|x|}\left(|x|\vee 1\right), \tag{62}$$

which, in particular, yields that $\partial_T\tilde{\alpha}_t(T,\cdot)\in\tilde{\mathcal{B}}^0$. The above estimate also shows integrability of $\partial_T\tilde{\alpha}_t$ as a mapping $[0,\bar{T}]\hookrightarrow\tilde{\mathcal{B}}^0$. And since, due to Hille's theorem (see Theorem II.6 in [17]), we can interchange the integration of a $\tilde{\mathcal{B}}^0$-valued function and the application of a continuous functional (notice that Dirac delta-function is a continuous functional on $\tilde{\mathcal{B}}^0$), we deduce that $\tilde{\alpha}_t(T)=\tilde{\alpha}_t(0)+\int_0^T\partial_u\tilde{\alpha}_t(u)du$, where the integral is understood as a Bochner integral of a $\tilde{\mathcal{B}}^0$-valued function. Therefore, we conclude that the actual derivative, $\frac{d}{dT}\tilde{\alpha}_t$, coincides with the partial derivative $\partial_T\tilde{\alpha}_t$.

Finally, estimates (58) and (62) complete the proof: $\tilde{\alpha}$ is a progressively measurable integrable random process of $t\in[0,\bar{T}]$, with values in $\tilde{\mathcal{B}}$. ∎

References

1. H. Buehler. Expensive martingales. *Quantitative Finance*, 6:207–218, 2006.
2. R. Carmona and S. Nadtochiy. An infinite dimensional stochastic analysis approach to local volatility dynamic models. *Communications on Stochastic Analysis*, 2(1):109 – 123, 2008.
3. R. Carmona and S. Nadtochiy. Local volatility dynamic models. *Finance and Stochastics*, 13(1):1 – 48, 2009.
4. R. Carmona and M. Tehranchi. *Interest Rate Models: an Infinite Dimensional Stochastic Analysis Perspective*. Springer Verlag, 2006.
5. P. Carr, H. Geman, D. Madan, and M. Yor. The fine structure of asset returns: An empirical investigation. *Journal of Business*, 75(2):305 – 332, 2002.
6. P. Carr, H. Geman, D. Madan, and M. Yor. From local volatility to local lévy models. *Quantitative Finance*, 4:581 – 588, 2005.
7. P. Carr and D. Madan. Option valuation using the fast fourier transform. *Journal of Computational Finance*, 2(5):61 – 73, 1998.
8. R. Cont and A. Bentata. Mimicking the marginal distributions of a semimartingale. *Preprint submitted to the Académie des Sciences*, 2009.
9. R. Cont, I. da Fonseca, and V. Durrleman. Stochastic models of implied volatility surfaces. *Economic Notes*, 31(2), 2002.
10. R. Cont and J. da Fonseca. Dynamics of implied volatility surfaces. *quantitative Finance*, 2:45–60, 2002.
11. R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman and Hall/CRC, 2004.
12. R. Cont and E. Voltchkova. Integro-differential equations for option prices in exponential lévy models. *Finance and Stochastics*, 9:299 – 325, 2005.
13. L. Cousot. Necessary and sufficient conditions for no-static arbitrage among european calls. Technical report, Courant Institute, New York University, Octobber 2004.
14. G. da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*. Cambridge University Press, 1992.
15. M.H.A. Davis and D.G. Hobson. The range of traded option prices. *Mathematical Finance*, 17:1–14, 2007.
16. E. Derman and I. Kani. Stochastic implied trees: Arbitrage pricing with stochastic term and strike structure of volatility. *International Journal of Theoretical and Applied Finance*, 1:61–110, 1998.

17. J. Diestel and J. Uhl. *Vector measures*. American Mathematical Society, 1979.
18. B. Dupire. Pricing with a smile. *Risk*, 7:32–39, 1994.
19. I. Gyöngy. Mimicking the one-dimensional marginal distributions of processes having an ito differential. *Probability Theory and Related Fields*, 71(4):501 – 516, 1986.
20. J. Jacod. Une généralisation des semimartingales: les processus admettant un processus à accroissements indépendants tangent.
21. J. Jacod and P. Protter. Risk neutral compatibility with option prices. Technical report, Université de Paris VI and Cornell University, April 2006.
22. J. Jacod and H. Sadi. Processus admettant un processus à accroissements indépendants : Cas général.
23. J. Jacod and A. Shiryaev. *Limit theorems for stochastic processes*. Springer-Verlag Berlin Heidelberg, 2003.
24. S. Kou. A jump-diffusion model for option pricing. *Management Science*, 48(8):1086 – 1101, 2002.
25. H. Kuo. *Gaussian measures in Banach spaces*. Springer-Verlag Berlin, 1975.
26. J.P. Laurent and D. Leisen. Building a consistent pricing model from observed option prices. World Scientific, 2000.
27. D. Lepingle and J. Mémin. Sur l'intégrabilité uniforme des martingales exponentielles. *Probability Theory and Related Fields*, 42(3):175 – 203, 1978.
28. D. Madan and E. Seneta. The variance gamma (v.g.) model for share market returns. *Journal of Business*, 63(4):511 – 24, 1990.
29. R. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144, 1976.
30. P. Protter. *Stochastic integration and differential equations*. Springer-Verlag Berlin Heidelberg, 2005.
31. K. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge University Press, 1999.
32. P. Schönbucher. A market model for stochastic implied volatility. *Phil. Trans. of the Royal Society, Series A*, 357:2071 – 2092, 1999.
33. M. Schweizer and J. Wissel. Arbitrage-free market models for option prices: The multi-strike case. *Finance and Stochastics*, 12(4):469–505, 2008.
34. M. Schweizer and J. Wissel. Term structures of implied volatilities: Absence of arbitrage and existence results. *Mathematical Finance*, 18(1):77–114, 2008.