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MONTE CARLO MALLIAVIN COMPUTATION OF THE SENSITIVITIES OF SOLUTIONS OF SPDEs*

RENÉ CARMONA† AND LIXIN WANG†

Abstract. This paper deals with an application of the Malliavin calculus to a stochastic partial differential equation of the Schrödinger type. This equation appears as the major building block in the analysis of the focusing properties of time-reversed waves in a random medium in the asymptotic regime where the parabolic approximation is valid. We consider the sensitivities of the solutions with respect to several parameters, and we provide closed form formulae in terms of Skorohod integrals with respect to an infinite dimensional Wiener process. We construct finite dimensional approximation schemes for these integrals. These schemes are based on a sieve of Wiener chaos expansions mixed with Galerkin approximations in a natural Fourier basis. In two space dimensions, our computational algorithm seems to perform better than those we found in the literature. Moreover, because it avoids finite difference methods, it can be implemented in three space dimensions without much ado.

Key words. time reversal, Malliavin calculus, parameter sensitivities

AMS subject classifications. 60H07, 60H15, 60B0, 60B5

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1. Introduction.

One of the most important steps in the analysis of a dynamical system is the computation of the dependencies of the state evolution upon initial conditions and other parameters of the problem. This calculus of variations has been extended to the case of stochastic dynamical systems where differentiation with respect to the initial condition leads to the Jacobian flow. The computation of these sensitivities is of crucial importance in financial applications. There, they provide effective risk management tools by giving the hedges needed to mitigate the risk associated with derivatives. Since the Malliavin calculus emerged out of the stochastic calculus of variations, it was no surprise to see it come to the rescue for the computation of the Greeks, which is the name commonly given to the sensitivities of the prices as given by expectations of derivative pay-offs. The ground-breaking works of Lions and his collaborators [10] and [11] were based on the use of Malliavin calculus to derive expressions amenable to efficient Monte Carlo computations. In the present study, we extend these ideas to the infinite dimensional case of stochastic partial differential equations.

Applications of the Malliavin calculus to infinite dimensional fixed income models were given in [5]. However, the spirit was different. There, Malliavin derivatives were used to extract information from the Clark–Ocone formula on the integrands appearing in various martingale representations as stochastic integrals. Here, we depart from the kind of financial problem resolved in [5]. Instead, we work in the spirit of the Monte Carlo approach introduced in [10], and in order to probe the efficiency of the tools we develop, we use an application from wave propagation in random media as a test bed. We consider models in which waves emitted from a point source, say the

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†Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544 (rcarmona@princeton.edu, lxwang@princeton.edu). The first author is also with the Bendheim Center for Finance and the Applied and Computational Mathematics Program.

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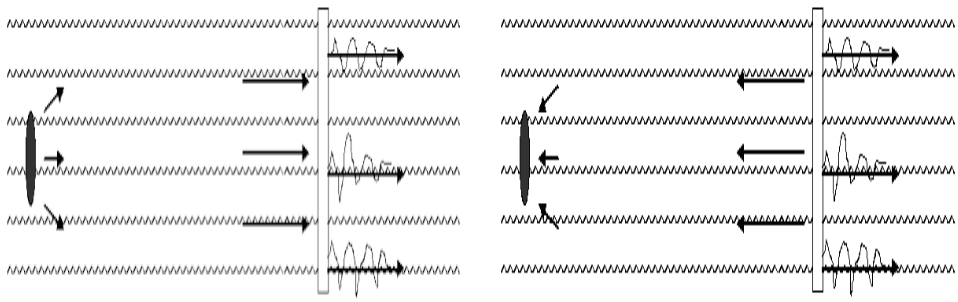

FIG. 1. *Schematic of a time reversal experiment amenable to the parabolic approximation used in this paper: outgoing field (left) and incoming field (right) after time reversal on the mirror.*

origin in $\mathbb{R}^3$, are time-reversed when they are detected on a set of receivers, and then re-emitted in the medium. We shall denote by A the region covered by these special receivers, which are called *mirrors* for obvious reasons. Figure 1 gives a schematic of such an experiment. As expected, the time-reversed signal refocuses at the source, but what is remarkable is the fact that this refocusing is enhanced by inhomogeneities in the medium. This enhanced refocusing was demonstrated experimentally by the physicist Mattias Fink, and the mathematical framework which we describe below was used by several authors to prove that the randomness of the medium was making the mirror region behave as though larger than it really is. See [2], [1], [15], and [16], for example. The original model involves the classical wave equation

$$\frac{1}{c(x,y,z)^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 0,$$

where $c(x,y,z)$ denotes the propagation speed at point (x,y,z) in the medium. Using the Fourier transform of the solution in the time variable,

$$\tilde{u}(\omega,x,y,z) = \int_{\mathbb{R}} e^{-i\omega t} u(t,x,y,z)\,dt,$$

the wave equation can be rewritten equivalently as one equation for each frequency, namely,

$$\Delta\tilde{u} + k^2 n(x,y,z)^2\tilde{u} = 0 \tag{1}$$

with

$$k = \frac{\omega}{c_0} \qquad \text{and} \qquad n(x,y,z) = \frac{c_0}{c(x,y,z)}.$$

Thus, the strategy is to solve this equation for each fixed k and then compute the inverse-Fourier transform. Each equation (1) is of the Helmholtz type. We solve them in the asymptotic regime of what is usually called the *parabolic approximation.* This asymptotic regime is introduced by the search for a solution of the form

$$\tilde{u}(k,x,y,z) = e^{ikz}\psi(k,x,y,z),$$

for in this case the function $\psi(k,x,y,z)$ needs to satisfy

$$2ik\partial_z\psi + \partial^2_{zz}\psi + \Delta_{\mathbf{x}}\psi + k^2[n(\mathbf{x},z)^2 - 1]\psi = 0$$

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with $\mathbf{x} = (x, y)$. In most applications, the size a of the mirror is much smaller than the distance L between the source and the mirror. In other words, $a \ll L$, a condition which gives validity to the so-called *narrow beam approximation.* In this approximation, we assume that the function ψ varies slowly in the z-direction (i.e., the direction of propagation between the pulse source and the mirror), which implies that $k|\partial_z\psi| \gg |\partial_{zz}\psi|$. Under this condition, the fixed-k Helmholtz equations can be rewritten in the form

$$\begin{cases} 2ik\partial_z\psi + \Delta_{\mathbf{x}}\psi + k^2(n(\mathbf{x}, z)^2 - 1)\psi = 0, \\ \psi(\mathbf{x}, z = 0) = \psi_0(\mathbf{x}). \end{cases}$$

This last form looks like a two dimensional Schrödinger equation in the space variable $\mathbf{x}$, the variable z playing the role of time, and the potential being random and time dependent! Most of the results of the present paper concern the analysis of this random partial differential equation.

We add several simplifying assumptions before we actually tackle the analysis of this partial differential equation. We restrict ourselves to situations in which the following hold:

- the wavelength λ of the initial pulse is short compared to the propagation distance L,

$$\epsilon \equiv \frac{\lambda}{L} \ll 1;$$

- the fluctuations of the index of refraction are weak and isotropic and satisfy

$$\mathbb{E}\{(n^2 - 1)\} = 0 \qquad \text{and} \qquad \mathbb{E}\{(n^2 - 1)^2\} = O(\epsilon);$$

- the correlation length ℓ of the fluctuations of the medium in the z-direction are of the same order of magnitude as the wavelength λ of the initial pulse,

$$\ell \sim \lambda.$$

Henceforth, we shall assume that the random potential is of the form

$$n(\mathbf{x}, z) = 1 + \sqrt{\epsilon}\mu(z, \mathbf{x})$$

for a small parameter ϵ. A weak convergence argument of the central limit theorem type can be used to show that, in the limit $\epsilon \to 0$, this random potential converges in distribution toward a mean zero Gaussian field which behaves like white noise in the variable z. This justifies our assuming that the random potential is a Gaussian white noise in time (remember that z is the time variable in our Schrödinger equation) with a smooth correlation in the remaining space dimensions.

A time-reversal experiment consists of detecting and reversing the signal detected on a mirror $A \subset \mathbb{R}^2 \times \{L\}$. So for each fixed time frequency k (i.e., after the Fourier transform in time of the wave field has been computed), a phase-conjugated version of the time-reversed wave field produced in this way on the plane $z = 0$ is given by

$$\psi^r(\mathbf{y}) = \int_A G(L, \mathbf{y}; \mathbf{x})\overline{\phi_0(\psi(L, \mathbf{x}))}d\mathbf{x}, \tag{2}$$

where ϕ_0 is a cut-off function introduced for the sake of regularization. In terms of the physics of the mirror, this cut-off function can be thought of as a way to model

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the smoothing effect of the receivers which detect only the frequency modes (of the Fourier transform in time of the incoming wave field) with a significant amplitude. For the sake of definitiveness we shall use $\phi_0(z) = z\mathbf{1}_{\{|z|>s_0\}}$ for some threshold s_0. In this formula, $\psi(z, \mathbf{x})$ denotes the solution of the stochastic Schrödinger equation with initial condition $\psi_0(\mathbf{x})$ given by the initial pulse, while $G(z, \mathbf{x}; \mathbf{y})$ denotes the Green's function, i.e., for fixed $\mathbf{y}$, the solution of the same stochastic Schrödinger equation with initial condition

$$G(z = 0, \mathbf{x}; \mathbf{y}) = \delta_{\mathbf{x}}(\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}).$$

The goal of the present study is to develop analytical and computational tools to analyze and compute the sensitivities of the time-reversed wave field with respect to a parameter of the problem, say γ. Possible values for this parameter γ include the parameters affecting the shape of the initial pulse ψ_0, the wave time frequency, or even the variance of the random medium fluctuations.

The weak convergence argument alluded to above to justify the parabolic approximation with a white noise in the propagation direction is incomplete. Indeed, its derivation requires that the time frequency ω (or equivalently the Fourier variable k) be fixed, and strictly speaking, one should wonder whether or not the white noise driving the parabolic equation (of Schrödinger type) should change with k. The fact that one can use the same white noise for all the frequencies was justified in [2] in the one dimensional (1D) case, and in [3] where the multifrequency 3D case is treated.

The parabolic approximation described above has proven to be a very efficient analysis tool when applicable. The interested reader is referred to [8] and [9] and the references therein for recent applications to different wave propagation problems.

We close this introduction with a short summary of the paper. Section 2 introduces the function spaces in which the stochastic Schrödinger equation is analyzed, and existence of solutions with square integrable initial conditions are proven together with the existence of the Green's function. Section 3 is devoted to the Malliavin differentiability of these solutions, while section 4 contains the computations of the partial derivatives of the time-reversed field with respect to parameters. Section 5 is devoted to the construction of numerical schemes to compute the solutions of the Schrödinger stochastic partial differential equation, the time-reversed field, and its sensitivities. Finally, section 6 reports results of numerical experiments, including comparisons of our numerical implementations in the 2D case with results found in the literature. The paper concludes with an appendix containing the statements, together with detailed proofs, of two technical results needed in the text.

2. Schrödinger SPDE of the parabolic approximation.

Motivated by the discussion above, we consider the bilinear stochastic partial differential equation (SPDE):

$$\frac{\partial\psi(t, \mathbf{x})}{\partial t} = \frac{i}{2k}\Delta\psi(t, \mathbf{x}) - \frac{i}{2}\psi(t, \mathbf{x})\dot{W}(t, \mathbf{x}), \tag{3}$$

over the region $D = [-l, l] \times [-l, l]$ with initial condition $\psi(0, \mathbf{x}) = \psi_0(\mathbf{x})$ for $\mathbf{x} = (x_1, x_2) \in D$. Here $\dot{W}$ denotes the derivative with respect to the time variable t. Note that the role of this time variable t was played by z in the description of the wave propagation experiment described earlier. We switch to the notation t for consistency with the standard literature on SPDEs. As a consequence, what was the distance L to the mirror in the transverse direction of z will be denoted by T in the following

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sections. We impose periodic boundary conditions by requiring that for all $t > 0$ we have

$$\left\{\begin{array}{l}\psi(t,-l,x_2)=\psi(t,l,x_2),\\ \psi(t,x_1,-l)=\psi(t,x_1,l),\\ \partial_1\psi(t,-l,x_2)=\partial_1\psi(t,l,x_2),\\ \partial_2\psi(t,x_1,-l)=\partial_2\psi(t,x_1,l),\end{array}\right. \tag{4}$$

where we used the notation $\partial_1\psi$ and $\partial_2\psi$ for the partial derivatives of ψ with respect to x_1 and x_2, respectively. The time dependent random potential of the Schrödinger equation appears as the time derivative of a bona fide Gaussian random field $\mathbf{W} = \{W(t,\mathbf{x}); t \geq 0, \mathbf{x} \in D\}$, which has mean zero and whose covariance form is given by

$$\begin{aligned}\Gamma_W(s,t;\phi,\varphi) &= \mathbb{E}[\langle\phi,W(t)\rangle\langle\varphi,W(s)\rangle\\ &= s\wedge t\iint_{\mathbf{x},\mathbf{y}\in D}\phi(\mathbf{x})\varphi(\mathbf{y})q(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}\end{aligned}$$

for some scalar function q. Here and throughout the paper we use the notation $\langle,\rangle$ and $\|\cdot\|$ for the inner product and the norm of the Hilbert space $L^2(D,d\mathbf{x})$, i.e.,

$$\langle f,g\rangle = \int_D f(\mathbf{x})\overline{g(\mathbf{x})}\,d\mathbf{x} \qquad \text{and} \qquad \|f\|^2 = \int_D |f(\mathbf{x})|^2\,d\mathbf{x}. \tag{5}$$

For the sake of definiteness, we denote by $(\Omega,\mathcal{F},\mathbb{P})$ the probability space on which the random field $\mathbf{W}$ is defined. Since $\mathbf{W}$ is a Wiener process in time, a limiting argument can be used to give a rigorous mathematical meaning to $\dot{W}(t,\mathbf{x})dt$, and the random Schrödinger equation (3) can be interpreted as a stochastic differential equation,

$$d\psi(t,\mathbf{x}) = \frac{i}{2k}\Delta\psi(t,\mathbf{x})dt - \frac{i}{2}\psi(t,\mathbf{x})\circ dW(t,\mathbf{x}), \tag{6}$$

of the Stratonovich type. This random Schrödinger equation was considered by Dawson and Papanicolaou in [7]. They constructed mild solutions by rewriting it in evolution form given by the stochastic integral equation

$$\psi(t) = U(t)\psi_0 - \frac{i}{2}\int_0^t U(t-s)\psi(s)W(ds), \tag{7}$$

where the stochastic integral is now of the Itô type and $\{U(t);\, t\in\mathbb{R}\}$ is the operator group solving the deterministic Schrödinger equation

$$\begin{aligned}\frac{\partial\psi(t,\mathbf{x})}{\partial t} &= \left[\frac{i}{2k}\Delta - \frac{1}{8}q(\mathbf{x},\mathbf{x})\right]\psi(t,\mathbf{x}),\\ \psi(0,\mathbf{x}) &= \psi_0(\mathbf{x}), \qquad \mathbf{x}\in D,\end{aligned}$$

with the same periodic boundary conditions (4).

We assume that the white noise is statistically homogeneous in the space variable $\mathbf{x}$. Spatial homogeneity of the noise, together with our choice of periodic boundary conditions, forces the covariance function $q(\mathbf{x},\mathbf{y})$ to be an even function (which we still denote q) of the difference $\mathbf{y}-\mathbf{x}$. Using the Fourier expansion of a periodic function with period $2l$, we get

$$q(\mathbf{x}) = \sum_{m,n=-\infty}^{\infty}\alpha_{m,n}B_{m,n}(\mathbf{x}), \qquad \mathbf{x}\in D,$$

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where $\alpha_{m,n}$'s are positive numbers (giving the spectral density of the spatial part of the white noise) which we assume to be summable in the sense that

$$\sum_{m,n=-\infty}^{\infty} \alpha_{m,n} < \infty. \tag{8}$$

Notice that $\sigma^2 = q(0)$ (which is equal to the sum of the above doubly infinite series divided by $2l$) measures the size of the fluctuations of the medium in the sense that $\mathbb{E}\{W(t,\mathbf{x})^2\} = t\sigma^2$. Here, $\{B_{m,n}\}_{m,n}$ is the Fourier orthonormal basis of $L^2(D, d\mathbf{x})$ given by

$$B_{m,n}(x_1,x_2) = \frac{1}{2l} e^{i\pi(mx_1+nx_2)/l}, \qquad -\infty < m,n < +\infty.$$

Notice that

$$[U(t)f](\mathbf{x}) = e^{-\sigma^2 t/8} \sum_{m,n=-\infty}^{\infty} e^{-i\lambda_{m,n}t/(2k)} \langle f, B_{m,n}\rangle \, B_{m,n}(\mathbf{x}),$$

where $\lambda_{m,n} = (m^2+n^2)\pi^2/l^2$ are the eigenvalues of the Laplacian on the domain D with periodic boundary conditions. Also,

$$\|U(t)f\|^2 = \langle U(t)f, U(t)f\rangle = e^{-\sigma^2 t/4} \, \|f\|^2 . \tag{9}$$

It is convenient to associate an integral operator, say Q, with the covariance function q by setting

$$[Qf](\mathbf{x}) = \int_D q(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y},$$

and, using the Fourier expansion of the covariance function q, we see that

$$[Qf](\mathbf{x}) = 2l \sum_{m,n=-\infty}^{\infty} \alpha_{m,n} \langle f, B_{m,n}\rangle B_{m,n}(\mathbf{x}).$$

We shall need a mild *technical assumption* concerning the existence of a nonnegative sequence $\{\beta_{m,n}\}_{m,n}$ such that $\sum_{m,n} \beta_{m,n} < \infty$, and most importantly,

$$\sum_{m',n'} \alpha_{m-m',n-n'} \beta_{m',n'} \le C\beta_{m,n}, \qquad m,n \in \mathbb{Z}, \tag{10}$$

for some constant $C > 0$. We prove in the appendix that this assumption is not restrictive since it is satisfied in most interesting cases.

2.1. Function spaces, Hilbert–Schmidt operators, and stochastic integrals. The need to use stochastic integrals with respect to a genuinely infinite dimensional Wiener process prompts us to introduce a scale of Hilbert spaces in which these integrations will take place. Because of the special properties of the Fourier basis,

$$f \in L^2(D, d\mathbf{x}) \Leftrightarrow \|f\|^2 = \sum_{m,n} \langle f, B_{m,n}\rangle^2 < \infty,$$

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and it is most convenient to define the scale of Hilbert spaces via expansions of functions and distributions in the Fourier basis. For each double sequence $a = \{a_{m,n}\}_{m,n}$ we define the space

$$H_a = \left\{ f;\ \sum_{m,n} a_{m,n}^2 \langle f, B_{m,n}\rangle^2 < \infty \right\}. \tag{11}$$

Equipped with the norm

$$\|f\|_{H_a}^2 = \sum_{m,n} a_{m,n}^2 \langle f, B_{m,n}\rangle^2,$$

the space H_a becomes a (separable) Hilbert space. Using the duality provided by the inner product (5) of $H_1 = L^2(D, d\mathbf{x})$, the dual of H_a can be identified with the space $H_a^* = H_{1/a}$ for $f \in H_a^*$ if and only if

$$\sum_{m,n} a_{m,n}^2 < f, B_{m,n} >^{-2} < \infty,$$

in which case the value of this sum gives the squared norm $\|f\|_{H_a^*}^2$. Notice that with this notation we have

$$\mathbb{E}\{\langle f, W(t)\rangle^2\} = t\langle Qf, f\rangle = t \sum_{m,n=-\infty}^{\infty} 2l\alpha_{m,n}\langle f, B_{m,n}\rangle^2 = 2lt\|f\|_{H_a}^2$$

for the sequence $a = \{a_{m,n}\}_{m,n}$ given by

$$a_{m,n} = \sqrt{\alpha_{m,n}}. \tag{12}$$

This shows that, with this particular choice (12) for the sequence a, the random potential field $\mathbf{W} = \{W(t, \mathbf{x}); t \geq 0, \mathbf{x} \in D\}$ can be viewed as the cylindrical Wiener process of the Hilbert space $H = H_{1/\sqrt{\alpha}}$. In other words, the Hilbert space $H = H_{1/\sqrt{\alpha}}$ can be viewed as the reproducing kernel Hilbert space of the infinite dimensional Wiener process $\{W(t)\}_t$. See, for example, [4] for details.

It is useful to compare the Hilbert spaces of this scale to the Hilbert space naturally associated with the quadratic form of the nonnegative operator Q defined above. This Hilbert space is defined via its norm,

$$\| f \|_Q^2 = \langle Qf, f\rangle,$$

so that the corresponding Hilbert space H_Q is given by

$$H_Q = \{f;\ \| f \|_Q^2 < \infty\}.$$

Hence, $H_Q = H_a$ for the sequence $a = \{a_{m,n}\}_{m,n}$ given by (12), so that $H_Q = H_{\sqrt{\alpha}} = H^*$.

Stochastic integrals $\int_0^t \sigma_s dW(s)$ can be defined as martingales with values in a Hilbert space F whenever $\sigma = \{\sigma_s\}_s$ is an adapted process with values in the space of Hilbert–Schmidt operators from H into a (possibly different) Hilbert space F satisfying

$$\mathbb{E}\left\{ \int_0^t \|\sigma_s\|_{\mathcal{L}_{HS}(H,F)}^2 \, ds \right\} < \infty, \tag{13}$$

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where, here and in the following, $\mathcal{L}_{HS}(H,F)$ denotes the space of Hilbert–Schmidt operators from H into F. Notice that, since $\{\sqrt{\alpha_{m,n}}B_{m,n}\}_{m,n}$ is an orthonormal basis of H, we have

$$\| U \|^2_{\mathcal{L}_{HS}(H,F)} = \sum_{m,n} \alpha_{m,n} \, \| U \overline{B_{m,n}} \|^2_F \tag{14}$$

whenever $U \in \mathcal{L}_{HS}(H,F)$.

LEMMA 1. *Let $\{f(t)\}_t$ be an adapted process taking values in a space $F = H_a$ for some nonnegative double sequence $a = \{a_{m,n}\}_{m,n}$ satisfying either*

(i) $\int_0^t \mathbb{E}\{\|f(s)\|^2\}\, ds < \infty$ *and* $\sup_{m,n} a_{m,n} < \infty$, *or*

(ii) $\int_0^t \mathbb{E}\{\sup_{m,n} |\langle f(s), B_{m,n}\rangle|^2\}\, ds < \infty$ *and* $\sum_{m,n} a^2_{m,n} < \infty$.

If we define the operator $\sigma_t(f)$ as the multiplication operator

$$H \ni h \hookrightarrow \sigma_t(f)h = f(t)h,$$

then in either case the stochastic integral

$$\int_0^t \sigma_s(f) W(ds) = \int_0^t f(s) W(ds)$$

makes sense as an element of F.

Proof. We just need to check that condition (13) holds. For each fixed $s > 0$ we have

$$\begin{aligned}
\|\sigma_s(f)\|^2_{\mathcal{L}_{HS}(H,F)} &= \sum_{m,n} \alpha_{m,n} \|f(s)\overline{B_{m,n}}\|^2_F \\
&= \sum_{m,n,m',n'} \alpha_{m,n} a^2_{m',n'} |\langle f(s)\overline{B_{m,n}}, B_{m',n'}\rangle|^2 \\
&= \frac{1}{2l} \sum_{m,n,m',n'} \alpha_{m,n} a^2_{m',n'} |\langle f(s), B_{m+m',n+n'}\rangle|^2 \\
&= \frac{1}{2l} \sum_{m,n,m",n"} \alpha_{m,n} a^2_{m"-m,n"-n} |\langle f(s), B_{m",n"}\rangle|^2,
\end{aligned}$$

and the right-hand side is always smaller than or equal to a constant times

$$\sum_{mn} |\langle f(s), B_{m,n}\rangle|^2 = \|f(s)\|^2_{L^2(D,d\mathbf{x})}$$

under condition (i) or

$$\sup_{mn} |\langle f(s), B_{m,n}\rangle|^2$$

under condition (ii). In either case, taking expectation and integrating over s, we conclude. ∎

We shall use the result of the above lemma repeatedly throughout the paper with $F = H_{\sqrt{\alpha}}$. Notice that, in this case, $a_{m,n} = \sqrt{\alpha_{m,n}}$ and both conditions on $a_{m,n}$ appearing in conditions (i) and (ii) are satisfied.

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2.2. Mild solutions of the evolution form. In the spirit of [7], we construct a mild solution by formal expansion in the Wiener chaos, identifying the hierarchy of equations satisfied by the projections of the solution onto the various chaos, solving these equations inductively, and summing them up to recover the desired solution. Wiener chaos expansion can be a powerful tool for solving bilinear SPDEs. See, for example, [6], [12], [13], [14]. For any initial condition ψ_0, we define recursively the terms of the Wiener chaos expansion as the sequence of processes $\{\psi^N(t);\, t \geq 0\}$ indexed by $N \geq 0$:

$$\psi^0(t) = U(t)\psi_0 \tag{15}$$

and

$$\psi^N(t) = -\frac{i}{2}\int_0^t U(t-s)\psi^{N-1}(s)W(ds), \qquad N > 0. \tag{16}$$

PROPOSITION 2. *For each $\psi_0 \in L^2(D, d\mathbf{x})$, the series $\sum_{N\geq 0}\psi^N(t)$ whose terms are defined by* (15) *and* (16) *converges toward an $L^2(D, d\mathbf{x})$-valued process $\{\psi(t);\, t \geq 0\}$ in the sense that*

$$\lim_{N\to\infty} \mathbb{E}\left\{\left\|\psi(t) - \sum_{N\geq 0}\psi^N(t)\right\|^2\right\} = 0 \tag{17}$$

and $\{\psi(t);\, t\geq 0\}$ solves the evolution equation (7) *in $L^2(D, d\mathbf{x})$.*

Proof. Using the properties of the Brownian motion W, we get

$$\begin{aligned}
&\mathbb{E}\{\langle\psi^N(t), \psi^N(t)\rangle\}\\
&= \frac{1}{4}\int_0^t\int_0^t \mathbb{E}\{\langle U(t-s_1)\psi^{N-1}(s_1)W(ds_1), U(t-s_2)\psi^{N-1}(s_2)W(ds_2)\rangle\}\\
&= \frac{1}{4}\int_{s_1,s_2\in[0,t]} e^{-\frac{\sigma^2(2t-s_1-s_2)}{8}} \sum_{(m,n),(m',n')=0}^{\infty} e^{-\frac{i}{2k}[\lambda_{m,n}(t-s_1)-\lambda_{m',n'}(t-s_2)]}\langle B_{m,n}, B_{m',n'}\rangle\\
&\quad\times \int_{\mathbf{x}_1,\mathbf{x}_2\in D}\overline{B_{m,n}(\mathbf{x}_1)}B_{m',n'}(\mathbf{x}_2)\\
&\qquad \mathbb{E}\{\psi^{N-1}(s_1,\mathbf{x}_1)\overline{\psi^{N-1}(s_2,\mathbf{x}_2)}W(ds_1,d\mathbf{x}_1)W(ds_2,d\mathbf{x}_2)\}\\
&\quad= \frac{1}{4}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)}\sum_{m,n=0}^{\infty}\int_{\mathbf{x}_1,\mathbf{x}_2\in D} q(\mathbf{x}_1-\mathbf{x}_2)\\
&\quad \mathbb{E}\{\psi^{N-1}(s,\mathbf{x}_1)\overline{\psi^{N-1}(s,\mathbf{x}_2)}\}\overline{B_{m,n}(\mathbf{x}_1)}B_{m,n}(\mathbf{x}_2)d\mathbf{x}_2 d\mathbf{x}_1 ds\\
&= \frac{1}{4}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)}2l\sum_{(m,n),(m',n')=0}^{\infty}\alpha_{m',n'}\mathbb{E}\{|\langle\psi^{N-1}(s)\overline{B_{m,n}}B_{m',n'}\rangle|\}ds\\
&= \frac{1}{4}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)}\sum_{(m,n),(m',n')=0}^{\infty}\frac{\alpha_{m',n'}}{2l}\mathbb{E}\{|\langle\psi^{N-1}(s)B_{m+m',n+n'}\rangle|\}ds\\
&= \frac{1}{4}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)}\sum_{(m',n')=0}^{\infty}\frac{\alpha_{m',n'}}{2l}\mathbb{E}\{\langle\psi^{N-1}\psi^{N-1}\rangle\}ds\\
&= \frac{\sigma^2}{4}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)}\mathbb{E}\{\langle\psi^{N-1}\psi^{N-1}\rangle\}ds.
\end{aligned}$$

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Repeating the same argument inductively to estimate the inner expectation starting with $\mathbb{E}\{<\psi^{N-1},\psi^{N-1}>\}$, and using

$$\begin{aligned}\|\psi^0(t)\|^2 &= \langle\psi^0(t),\psi^0(t)\rangle\\ &= \langle U(t)\psi_0 U(t)\psi_0\rangle\\ &= e^{-\sigma^2 t/4}\,\|\psi_0\|^2,\end{aligned}$$

we conclude that for each $t \geq 0$ and for each $N \geq 0$ we have

$$\mathbb{E}[\|\psi^N(t)\|^2] \leq \frac{1}{N!}\left(\frac{\sigma^2 t}{4}\right)^N e^{-\sigma^2 t/4}\,\|\psi_0\|^2,$$

and a fortiori

$$\mathbb{E}\{\|\psi^N(t)\|_Q^2\} \leq \sigma^2\mathbb{E}\{\|\psi^N(t)\|^2\} \leq \sigma^2\frac{1}{N!}\left(\frac{\sigma^2 t}{4}\right)^N e^{-\sigma^2 t/4}\,\|\psi_0\|^2\,.$$

Therefore, if $\|\psi_0\|^2<\infty$, for each $T>0$ the series $\sum_N \psi^N(t)$ converges in the Hilbert spaces $L^2([0,T]\times\Omega;K)$ of K-valued square integrable processes for $K = L^2(D,d\mathbf{x})$ and $K = H^*$. Moreover, summing up both sides of (16) in K gives that the infinite sum is a solution of the evolution equation. ∎

2.3. Existence of the Green's function. Formula (2) shows the need for the Green's function in order to analyze time-reversal in the parabolic approximation. The Green's function is nothing but the solution of the same PDE with a point mass (Dirac measure) as initial condition. Unfortunately, point masses do not belong to the function space $L^2(D,d\mathbf{x})$, and a special treatment is needed to show that one can solve the stochastic Schödinger equation (6) with a Dirac point mass as initial condition.

PROPOSITION 3. *For each ψ_0 satisfying*

$$\sup_{m,n}|\langle\psi_0,B_{m,n}\rangle|^2<\infty \tag{18}$$

the series $\sum_N\psi^N(t)$ converges toward an H^*-valued process $\{\psi(t);\, t\geq 0\}$ which satisfies the same estimates and which solves the evolution equation in $H_Q = H^*$.

Proof. For every $m,n>0$, we have

$$\begin{aligned}&\mathbb{E}\{|\langle\psi^N(t),B_{m,n}\rangle|^2\}\\ &= \frac{1}{4}\mathbb{E}\Bigg\{\Bigg|\sum_{m',n'=0}^{\infty}\int_0^t e^{-\frac{\sigma^2}{8}(t-s)}e^{-\frac{i}{2k}\lambda_{m',n'}(t-s)}\\ &\qquad \langle\psi^{N-1}(s)W(ds),B_{m',n'}\rangle\langle B_{m',n'},B_{m,n}\rangle\Bigg|^2\Bigg\}\\ &= \frac{1}{4}\mathbb{E}\left\{\left|\int_0^t e^{-\frac{\sigma^2}{8}(t-s)}e^{-\frac{i}{2k}\lambda_{m,n}(t-s)}\langle\psi^{N-1}(s)W(ds),B_{m,n}\rangle\right|^2\right\}\\ &= \frac{1}{4}\mathbb{E}\Bigg\{\int_0^t e^{-\frac{\sigma^2}{4}(t-s)}\int_{\mathbf{x}_1,\mathbf{x}_2\in D}q(\mathbf{x}_1-\mathbf{x}_2)\psi^{N-1}(s,\mathbf{x}_1)\\ &\qquad \overline{\psi^{N-1}(s,\mathbf{x}_2)B_{m,n}(\mathbf{x}_1)}B_{m,n}(\mathbf{x}_2)d\mathbf{x}_1 d\mathbf{x}_2 ds\Bigg\}\end{aligned}$$

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$$= \frac{2l}{4}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)} \sum_{m',n'=-\infty}^{\infty} \alpha_{m',n'} \mathbb{E}\left\{\left|\int_{\mathbf{x}\in D} \psi^{N-1}(s,\mathbf{x})\overline{B_{m,n}(\mathbf{x})B_{m',n'}(\mathbf{x})}d\mathbf{x}\right|^2\right\} ds.$$

Because

$$\left|\int_{\mathbf{x}\in D} \psi^{N-1}(s,\mathbf{x})\overline{B_{m,n}(\mathbf{x})B_{m',n'}(\mathbf{x})}d\mathbf{x}\right|^2 = \frac{1}{(2l)^2}\left|\int_{\mathbf{x}\in D} \psi^{N-1}(s,\mathbf{x})\overline{B_{m+m',n+n'}(\mathbf{x})}d\mathbf{x}\right|^2$$

we have

$$\begin{aligned}
&\sup_{m,n} \ \mathbb{E}\{|\langle \psi^N(t), B_{m,n}\rangle|^2\} \\
&\qquad \le \frac{1}{8l}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)} \sum_{m',n'=-\infty}^{\infty} \alpha_{m',n'} \sup_{m,n} \mathbb{E}\{|\langle \psi^{N-1}(s), B_{m+m',n+n'}\rangle|^2\} ds \\
&\qquad = \frac{1}{8l}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)} \sum_{m',n'=-\infty}^{\infty} \alpha_{m',n'} \sup_{m,n} \mathbb{E}\{|\langle \psi^{N-1}(s), B_{m,n}\rangle|^2\} ds \\
&\qquad = \frac{\sigma^2}{4}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)} \sup_{m,n} \mathbb{E}[|\langle \psi^{N-1}(s), B_{m,n}\rangle|^2] ds.
\end{aligned}$$

Note that

$$\begin{aligned}
\sup_{m,n} |\langle \psi^0(t), B_{m,n}\rangle|^2 &= \sup_{m,n} |\langle U(t)\psi_0, B_{m,n}\rangle|^2 \\
&= \sup_{m,n} |e^{-\sigma^2 t/8} e^{-\frac{i}{2k}\lambda_{m,n}(t)} \langle \psi_0, B_{m,n}\rangle|^2 \\
&= e^{-\sigma^2 t/4} \sup_{m,n} |\langle \psi_0, B_{m,n}\rangle|^2;
\end{aligned}$$

hence for any $N \ge 0$,

$$\begin{aligned}
\mathbb{E}\{\| \psi^N(t) \|_Q^2\} &\le 2l\sigma^2 \sup_{m,n} \mathbb{E}\{|\langle \psi^N(t), B_{m,n}\rangle|^2\} \\
&\le \frac{2l\sigma^2}{N!}\left(\frac{\sigma^2 t}{4}\right)^N e^{-\sigma^2 t/4} \sup_{m,n} |\langle \psi_0 B_{m,n}\rangle|^2.
\end{aligned}$$

Using the same argument as before, we derive the convergence of the infinite series $\sum_N \psi^N(t)$ and the existence of a solution to the evolution form of the random Schrödinger equation in $H_Q = H^*$ under condition assumption (18). ∎

We can now apply this result to the point source $\delta_{\mathbf{x}_0}$ for any $\mathbf{x}_0 \in D$. Indeed, if $\psi_0(\mathbf{x}) = \delta_{\mathbf{x}_0}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0)$, then

$$\sup_{m,n} |\langle \psi_0, B_{m,n}\rangle|^2 = \sup_{m,n} |B_{m,n}(\mathbf{x}_0)|^2 = \frac{1}{4l^2} < \infty,$$

which shows that the assumption of Proposition 3 holds. The corresponding solution of the stochastic Schrödinger equation will be denoted by $G(t, \mathbf{x}; \mathbf{x}_0)$ and will be called the Green's function of the problem.

3. Malliavin derivatives. The computation of sensitivities is based on the calculus of variations. Since we are working with SPDEs, we need a form of calculus of

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variations *in path space*. This is exactly what the Malliavin calculus was developed for. In this section, we prove the following result.

PROPOSITION 4. *The solution $\psi(t)$ of the Schrödinger SPDE* (6) *is differentiable in the sense of Malliavin, and its Malliavin derivative $D_s\psi(t)$ belongs to the space $\mathcal{L}_{HS}(H,F)$ of Hilbert–Schmidt operators from H into $F = H_Q = H^*$ whenever $\psi_0 \in L^2(D, d\mathbf{x})$ or ψ_0 satisfies the assumption* (18) *of Proposition* 3.

Proof. For the sake of convenience, we introduce a special notation for the partial sum approximations of the chaos expansion of the solution of the evolution equation:

$$\psi_N(t,\mathbf{x}) = \sum_{i=0}^{N} \psi^i(t,\mathbf{x}),$$

for $N \geq 0$. These partial sums satisfy the Picard iteration equations

$$\psi_{N+1}(t) = U(t)\psi_0 - \frac{i}{2}\int_0^t U(t-s)\psi_N(s)W(ds), \tag{19}$$

and from the computations of subsection 2.2 we derive the following inequalities:

$$\mathbb{E}\{\| \psi_N(t) \|^2\} \leq \| \psi_0 \|^2$$

and

$$\sup_{m,n} \mathbb{E}[|\langle \psi_N(t), B_{m,n}\rangle|^2] \leq \sup_{m,n} |\langle \psi_0, B_{m,n}\rangle|^2.$$

Both sides of (19) are differentiable in the sense of Malliavin, and their Malliavin derivatives satisfy the Picard-like iteration equation

$$D_s\psi_{N+1}(t) = -\frac{i}{2}U(t-s)\sigma_s(\psi_N) - \frac{i}{2}\int_s^t U(t-v)D_s\psi_N(v)W(dv), \qquad 0 \leq s \leq t.$$

See, for example, [4] or [5]. Note that $D_s\psi_N(t) \in \mathcal{L}_{HS}(H,F)$, so $D_s\psi_N(t)\overline{B_{m,n}} \in F$ satisfies

$$D_s\psi_{N+1}(t)\overline{B_{m,n}} = -\frac{i}{2}U(t-s)\psi_N(s)\overline{B_{m,n}} - \frac{i}{2}\int_s^t U(t-v)D_s\psi_N(v)\overline{B_{m,n}}W(dv).$$

Hence

$$\mathbb{E}\{\| D_s\psi_{N+1}(t)\overline{B_{m,n}} \|^2\} \leq \frac{1}{4}\mathbb{E}\{\| \psi_N(s)\overline{B_{m,n}} \|^2\} + \frac{\sigma^2}{4}\int_s^t \mathbb{E}\{\| D_s\psi_N(v)\overline{B_{m,n}} \|^2\}dv.$$

Multiplying both sides by $\alpha_{m,n}$ and summing over m and n, we get

$$\begin{aligned}
&\mathbb{E}\left\{\sum_{m,n} \alpha_{m,n} \| D_s\psi_{N+1}(t)\overline{B_{m,n}} \|^2\right\} \\
&\qquad \leq \frac{1}{4}\mathbb{E}\{\| \psi_N(s) \|^2\} + \frac{\sigma^2}{4}\int_s^t \mathbb{E}\left\{\sum_{m,n} \alpha_{m,n} \| D_s\psi_N(v)\overline{B_{m,n}} \|^2\right\} dv \\
&\qquad \leq \frac{1}{4} \| \psi_0 \|^2 + \frac{\sigma^2}{4}\int_s^t \mathbb{E}\left\{\sum_{m,n} \alpha_{m,n} \| D_s\psi_N(v)\overline{B_{m,n}} \|^2\right\} dv.
\end{aligned}$$

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Using the characterization (14) of the Hilbert–Schmidt norms in our scale of Hibert spaces setting, this inequality can be rewritten as

$$\mathbb{E}\left\{\| D_s\psi_N(t)\|^2_{\mathcal{L}_{HS}(H,F)}\right\} \le \frac{1}{4}\,\|\psi_0\|^2 + \frac{\sigma^2}{4}\int_s^t \mathbb{E}\{\| D_s\psi_N(v)\|^2_{\mathcal{L}_{HS}(H,F)}\,dv,$$

and using Gronwall's inequality, we get

$$\mathbb{E}\left\{\| D_s\psi_N(t)\|^2_{\mathcal{L}_{HS}(H,F)}\right\} \le \frac{\sigma^2}{4}\,\|\psi_0\|^2\, e^{\frac{\sigma^2}{4}(t-s)},$$

for any $N \ge 1$. So, according to Lemma 4.3 of [5], the mild solution $\psi(t)$ has Malliavin derivative whenever $\|\psi_0\|^2 < \infty$. Similarly, we have

$$\begin{aligned}
&\langle D_s\psi_{N+1}(t)\overline{B_{m,n}}, B_{m',n'}\rangle\\
&= -\frac{i}{2}\langle U(t-s)\psi_N(s)\overline{B_{m,n}}, B_{m',n'}\rangle - \frac{i}{2}\left\langle \int_s^t U(t-v)D_s\psi_N(v)\overline{B_{m,n}}W(dv), B_{m',n'}\right\rangle\\
&= -\frac{i}{4l}\langle U(t-s)\psi_N(s), B_{m+m',n+n'}\rangle - \frac{i}{2}\left\langle \int_s^t U(t-v)D_s\psi_N(v)\overline{B_{m,n}}W(dv), B_{m',n'}\right\rangle.
\end{aligned}$$

Following the arguments used in the proof of the existence of the evolution solution given in subsection 2.2 under the condition $\sup_{m,n}|\langle\psi_0, B_{m,n}\rangle|^2 < \infty$, we obtain

$$\begin{aligned}
&4\sup_{m',n'}\mathbb{E}\left\{|\langle D_s\psi_{N+1}(t)\overline{B_{m,n}}, B_{m',n'}\rangle|^2\right\}\\
&\le \frac{1}{4l^2}\sup_{m',n'}\left\{\mathbb{E}\left\{|\langle U(t-s)\psi_N(s), B_{m+m',n+n'}\rangle|^2\right.\right.\\
&\qquad\qquad \left.\left. + \left|\left\langle \int_s^t U(t-v)D_s\psi_N(v)\overline{B_{m,n}}W(dv), B_{m',n'}\right\rangle\right|^2\right\}\right\}\\
&\le \frac{1}{4l^2}e^{-\frac{\sigma^2}{4}(t-s)}\sup_{m',n'}\mathbb{E}\{|\langle\psi_N(s), B_{m',n'}\rangle|^2\}\\
&\qquad + \sigma^2\int_s^t e^{-\frac{\sigma^2}{4}(t-v)}\sup_{m',n'}\mathbb{E}\{|\langle D_s\psi_N(v)\overline{B_{m,n}}, B_{m',n'}\rangle|^2\}dv\\
&\le \frac{1}{4l^2}\sup_{m',n'}\mathbb{E}\{|\langle\psi_N(s), B_{m',n'}\rangle|^2\} + \sigma^2\int_s^t \sup_{m',n'}\mathbb{E}\{|\langle D_s\psi_N(v)\overline{B_{m,n}}, B_{m',n'}\rangle|^2\}dv.
\end{aligned}$$

Because

$$\sup_{m',n'}\mathbb{E}\{|\langle\psi_N(t), B_{m',n'}\rangle|^2\} \le \sup_{m',n'}|\langle\psi_0, B_{m',n'}\rangle|^2.$$

Using Gronwall's inequality again, we get

$$\sup_{m',n'}\mathbb{E}\{|\langle D_s\psi_N(t)\overline{B_{m,n}}, B_{m',n'}\rangle|^2\} \le \frac{1}{16l^2}\sup_{m',n'}|\langle\psi_0, B_{m',n'}\rangle|^2 e^{\sigma^2(t-s)/4},$$

and consequently

$$\begin{aligned}
\mathbb{E}[\| D_s\psi_N(t)\|^2_{\mathcal{L}_{HS}(H,F)}] &\le \sum_{m,n}\alpha_{m,n}\sup_{m',n'}\mathbb{E}\{|\langle D_s\psi_N(t)\overline{B_{m,n}}, B_{m',n'}\rangle|^2\}\\
&\le \frac{\sigma^2}{8l}\sup_{m',n'}|\langle\psi_0, B_{m',n'}\rangle|^2 e^{\sigma^2(t-s)/4}.
\end{aligned}$$

We conclude that the mild solution $\psi(t)$ is also Malliavin differentiable when the initial condition satisfies (18). $\square$

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4. Sensitivity analysis. In order to compute the derivatives of the time-reversed field intensity with respect to a parameter, we use the Malliavin differentiability of the mild solution of the parabolic approximation together with the chain rule. So, in the following discussion, we assume that the initial condition ψ_0 depends upon a parameter γ and satisfies

$$\| \psi_0 \|^2 < \infty \quad \text{and} \quad \sup_{m,n} |\langle \psi_0, B_{m,n} \rangle|^2 < \infty.$$

According to the results of the previous section, this guarantees existence and Malliavin differentiability of $\psi(t)$. Since $D_s\psi(t)$ is an operator mapping H into $F = H_Q = H^*$, we can apply it to the indicator function 1_D. We get

$$\begin{aligned} D_s\psi(t)1_D &= U(t-s)\left(-\frac{i}{2}\psi(s)\right) - \frac{i}{2}\int_s^t U(t-v)D_s\psi(v)1_D W(dv) \\ &= -\frac{i}{2}\psi(t) \end{aligned}$$

because of the uniqueness of mild solutions. As explained in the introduction, we model the physical limitations of the mirror device by the introduction of an amplitude threshold under which the wave arriving on the mirror A is ignored. In other words, for each fixed frequency k, we consider the function $\psi^r(\mathbf{y})$ whose definition was given in (2) and which we recall for the sake of convenience:

$$\psi^r(\mathbf{y}) = \int_A G(T, \mathbf{y}; \mathbf{x})\overline{\phi_0(\psi(T, \mathbf{x}))}d\mathbf{x},$$

where $\phi_0(z) = z\mathbf{1}_{|z|>s_0}$. Recall that A is the mirror, i.e., the domain representing the set of receivers, and that the time horizon T plays the role of the distance L to the mirror in the transverse direction.

4.1. Sensitivity with respect to $\boldsymbol{\gamma}$. In this subsection, we compute the sensitivity analysis of the mean fixed frequency reversed field $\mathbb{E}\{\psi^r(\mathbf{y})\}$ with respect to a parameter γ in the initial condition.

Proposition 5. *Let us assume that the initial one-frequency field ψ_0 depends upon a parameter γ in a smooth way. Then we have*

(20)
$$\frac{d}{d\gamma}\mathbb{E}\{\psi^r(\mathbf{y})\} = \mathbb{E}\left\{\left(\frac{1}{T}W(T,D) + \frac{i}{2}\right)\int_{\mathbf{x}\in A}\overline{\phi_0(\psi(T,\mathbf{x}))\beta_\gamma(T,\mathbf{x})}G(T,\mathbf{y};\mathbf{x})d\mathbf{x}\right\},$$

where $W(T,D) = \langle \mathbf{1}_D, W(T)\rangle = \int_0^T \int\int_D \dot{W}(t,\mathbf{x})\,dt\,d\mathbf{x}$ and β_γ is defined by

$$\beta_\gamma(T,\mathbf{x}) = \begin{cases} 2i\psi(T,\mathbf{x})^{-1}\frac{d\psi(T,\mathbf{x})}{d\gamma} & \text{if } \psi(T,\mathbf{x}) \neq 0, \\ 0 & \text{if } \psi(T,\mathbf{x}) = 0. \end{cases} \tag{21}$$

Proof. For each integer m we pick a smooth function ϕ_m such that

$$\phi_m(z) = \begin{cases} 0 & \text{if } |z| \leq s_0 - \frac{1}{m}, \\ z & \text{if } |z| > s_0, \end{cases}$$

and we extend it for $|z| \in [s_0 - \frac{1}{m}, s_0]$ in order to be continuously differentiable on the complex plane. By analogy with formula (2) we define the approximation

$$\psi^r_m(\mathbf{y}) = \int_A G(T,\mathbf{y};\mathbf{x})\overline{\phi_m(\psi(T,\mathbf{x}))}d\mathbf{x},$$

from which it follows that the derivative with respect to γ is given by

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$$\frac{d}{d\gamma}\mathbb{E}\{\psi^r_m(\mathbf{y})\} = \int_A \mathbb{E}\left\{G(T,\mathbf{y};\mathbf{x})\overline{\phi'_m(\psi(T,\mathbf{x}))\frac{d\psi(T,\mathbf{x})}{d\gamma}}d\mathbf{x}\right\}.$$

Here

$$\frac{d\psi(T)}{d\gamma} = Y_{0,T}\frac{d\psi_0}{d\gamma},$$

where $Y_{s,t}$ is a strong random operator in the sense of Skorohod satisfying, for $s \in [0,t]$,

$$Y_{s,t} = -\frac{i}{2}U(t-s) - \frac{i}{2}\int_s^t U(t-v)D_s\psi(v)W(dv).$$

Since $\phi'_m(\psi(T,\mathbf{x})) = 0$ whenever $\psi(T,\mathbf{x}) = 0$, the definition (21) of β_γ makes sense, and we have (fixing $\mathbf{x} \in A$ and $\mathbf{y} \in D$)

$$\begin{aligned}
&\mathbb{E}\left\{G(T,\mathbf{y};\mathbf{x})\overline{\frac{d\phi_m(\psi(T,\mathbf{x}))}{d\gamma}}\right\}\\
&\quad= \mathbb{E}\left\{\frac{1}{T}\int_0^T \overline{\phi'_m(\psi(T,\mathbf{x}))(D_t\psi(T)1_D)(\mathbf{x})}G(T,\mathbf{y};\mathbf{x})\overline{\beta_\gamma(T,\mathbf{x})}dt\right\}\\
&\quad= \mathbb{E}\left\{\frac{1}{T}\int_0^T \overline{D_t\phi_m(\psi(T))\overline{G(T,\mathbf{y};\mathbf{x})}\beta_\gamma(T,\mathbf{x})1_D(\mathbf{x})dt}\right\}\\
&\quad= \mathbb{E}\left\{\frac{\overline{\phi_m(\psi(T,\mathbf{x}))}}{T}\int_0^T (G(T,\mathbf{y};\mathbf{x})\overline{\beta_\gamma(T,\mathbf{x})}1_D)^*W(dt)\right\}.
\end{aligned}$$

The desired result follows because

$$\begin{aligned}
\frac{d}{d\gamma}\mathbb{E}\{\psi^r(\mathbf{y})\} &= \lim_{m\to\infty}\frac{d}{d\gamma}\mathbb{E}\{\psi^r_m(\mathbf{y})\}\\
&= \mathbb{E}\left\{\frac{1}{T}\int_{\mathbf{x}\in A}\overline{\phi_0(\psi(T,\mathbf{x}))}\int_0^T (G(T,\mathbf{y};\mathbf{x})\overline{\beta_\gamma(T,\mathbf{x})}1_D)^*W(dt)d\mathbf{x}\right\}.
\end{aligned}$$

The stochastic integral

$$\int_0^T (G(T,\mathbf{y};\mathbf{x})\overline{\beta_\gamma(T,\mathbf{x})}1_D)^*W(dt)$$

is not a stochastic integral in the usual Wiener sense. It is a Skorohod integral. We compute it as follows:

$$\begin{aligned}
&\mathbb{E}\left\{\frac{1}{T}\int_{\mathbf{x}\in A}\overline{\phi_0(\psi(T,\mathbf{x}))}\int_0^T (G(T,\mathbf{y};\mathbf{x})\overline{\beta_\gamma(T,\mathbf{x})}1_D)^*W(dt)d\mathbf{x}\right\}\\
&= \mathbb{E}\Bigg\{\frac{1}{T}\int_{\mathbf{x}\in A}\overline{\phi_0(\psi(T,\mathbf{x}))}\Bigg[G(T,\mathbf{y};\mathbf{x})\overline{\beta_\gamma(T,\mathbf{x})}\int_0^T (1_D)^*W(dt)\\
&\quad - \int_0^T D_t(G(T,\mathbf{y};\mathbf{x})\overline{\beta_\gamma(T,\mathbf{x})})1_D dt\Bigg]d\mathbf{x}\Bigg\}\\
&= \mathbb{E}\Bigg\{\frac{1}{T}\int_{\mathbf{x}\in A}\overline{\phi_0(\psi(T,\mathbf{x}))}\Bigg[G(T,\mathbf{y};\mathbf{x})\overline{\beta_\gamma(T,\mathbf{x})}(1_D)^*W(T) - \overline{\beta(T,\mathbf{x})}\int_0^T D_tG(T,\mathbf{y};\mathbf{x})1_D dt\\
&\qquad\qquad - G(T,\mathbf{y};\mathbf{x})\int_0^T \overline{D_t\beta_\gamma(T,\mathbf{x})1_D}dt\Bigg]d\mathbf{x}\Bigg\}.
\end{aligned}$$

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Since

$$D_t G(T,\mathbf{y};\mathbf{x})1_D = -\frac{i}{2}G(T,\mathbf{y};\mathbf{x}),$$

when $\psi(T,\mathbf{x}) \neq 0$, we have

$$\begin{aligned}
D_t\beta_\gamma(T,\mathbf{x})1_D &= -2i\frac{1}{\psi(T,\mathbf{x})^2}\frac{d\psi(T,\mathbf{x})}{d\gamma}D_t\psi(T,\mathbf{x})1_D + 2i\psi(T,\mathbf{x})^{-1}D_t\left(\frac{d\psi(T,\mathbf{x})}{d\gamma}\right)1_D \\
&= -\psi(T,\mathbf{x})^{-1}\frac{d\psi(T,\mathbf{x})}{d\gamma} + \psi(T,\mathbf{x})^{-1}\frac{d\psi(T,\mathbf{x})}{d\gamma} \\
&= 0,
\end{aligned}$$

and since $\phi_0(\psi(T,\mathbf{x})) = 0$ whenever $\psi(T,\mathbf{x}) = 0$, we get

$$\begin{aligned}
\frac{d}{d\gamma}\mathbb{E}\{\psi^r(\mathbf{y})\} &= \mathbb{E}\left\{\frac{1}{T}\int_{\mathbf{x}\in A}\overline{\phi_0(\psi(T,\mathbf{x}))\beta_\gamma(T,\mathbf{x})}G(T,\mathbf{y};\mathbf{x})\left((1_D)^*W(T) + \frac{i}{2}T\right)d\mathbf{x}\right\} \\
&= \mathbb{E}\left\{\left(\frac{1}{T}(1_D)^*W(T) + \frac{i}{2}\right)\int_{\mathbf{x}\in A}\overline{\phi_0(\psi(T,\mathbf{x}))\beta_\gamma(T,\mathbf{x})}G(T,\mathbf{y};\mathbf{x})d\mathbf{x}\right\},
\end{aligned}$$

which is the desired result. $\square$

4.2. Sensitivity with respect to k. The computation of the derivative of $\mathbb{E}\{\psi^r(\mathbf{y})\}$ with respect to the frequency k follows exactly the same lines, so we only sketch the proof.

PROPOSITION 6. *The derivative of the mean reversed one-frequency wave with respect to the frequency is given by*

$$\begin{aligned}
\mathbb{E}\bigg\{&\int_{\mathbf{x}\in A}\frac{dG(T,\mathbf{y};\mathbf{x})}{dk}\overline{\phi_0(\psi(T,\mathbf{x}))}d\mathbf{x} \\
&+\left(\frac{1}{T}W(T,D) + \frac{i}{2}\right)\int_{\mathbf{x}\in A}\overline{\phi_0(\psi(T,\mathbf{x}))\beta_k(T,\mathbf{x})}G(T,\mathbf{y};\mathbf{x})d\mathbf{x}\bigg\},
\end{aligned}$$

where $\beta_k = \beta_\gamma$ is as before.

Proof. Following the same argument as before, we get

$$\begin{aligned}
&\frac{d}{dk}\mathbb{E}\{\psi^r(\mathbf{y})\} \\
&= \int_{\mathbf{x}\in A}\mathbb{E}\bigg\{\frac{dG(T,\mathbf{y};\mathbf{x})}{dk}\overline{\phi_0(\psi(T,\mathbf{x}))} \\
&\qquad + \frac{1}{T}\overline{\phi_0(\psi(T,\mathbf{x}))}\int_0^T (G(T,\mathbf{y};\mathbf{x})\overline{\beta_k(T,\mathbf{x})}1_D)^*W(dt)\bigg\}d\mathbf{x},
\end{aligned}$$

where

$$\beta_k(T,\mathbf{x}) = \begin{cases} 2i\psi(T,\mathbf{x})^{-1}\frac{d\psi(T,\mathbf{x})}{dk} & \text{if } \psi(T,\mathbf{x}) \neq 0, \\ 0 & \text{if } \psi(T,\mathbf{x}) = 0 \end{cases}$$

and

$$\begin{aligned}
\frac{d\psi(t)}{dk} &= -\frac{i}{2}\int_0^t U(t-s)\frac{d\psi(s)}{dk}W(ds) \\
&\quad -\frac{i}{2}\int_0^t \frac{dU(t-s)}{dk}\psi(s)W(ds) + \frac{dU(t)}{dk}\psi_0.
\end{aligned}$$

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Replacing ψ_0 with $\delta_{\mathbf{x}_0}$, we see that $dG(T,\cdot;\mathbf{x}_0)/dk$ also satisfies the above equation. Computing the Skorohod integral as above gives the desired result. ∎

In the same way, we could also compute the sensitivity of the expected reverted wave with respect to the variance σ^2 of the white noise. Since this parameter enters in a linear way as a multiplicative factor in front of W, the computations would be similar. We refrain from giving the details, as the strategy for the derivation is exactly the same.

5. Numerical schemes. Since the 1D Fourier transform in time is easy to implement numerically, we discuss only numerical approximations of the solution of the Schrödinger equation for k fixed. For this equation, we propose a numerical scheme based on Fourier–Galerkin expansions of the solutions. This choice is imposed on us by the fact that the Fourier basis diagonalizes both the evolution operators $U(t)$ and the covariance integral operator Q. We use the superscript N to label the chaos expansion and the subscript M to label the projection in the Fourier–Galerkin approximation.

We proceed to a clear definition of our Fourier–Galerkin approximation scheme. For each positive integer M, we introduce the notation

$$I_M = \{(m,n) \in \mathbb{Z}\times\mathbb{Z};\ |m| \le M \text{ and } |n| \le M\},$$

and we define the operator $U_M(t)$ by

$$[U_M(t)f](\mathbf{x}) = e^{-\sigma^2 t/8} \sum_{m,n\in I_M} e^{-i\lambda_{m,n}t/(2k)} <f, B_{m,n}> B_{m,n}(\mathbf{x}).$$

For each M we define an approximate Nth chaos projection process $\psi_M^N(t)$ by $\psi_M^0(t) = U_M(t)\psi_0$ for $N=0$, and inductively for $N \ge 1$ by

$$\psi_M^N(t) = -\frac{i}{2}\int_0^t U_M(t-s)\psi_M^{N-1}(s)W(ds), \qquad N = 1,\dots,N_0.$$

As before, we prove convergence of the approximations in two different cases.

5.1. Numerical schemes when the initial condition is in $L^2(D, d\mathbf{x})$. We first consider the case $\psi_0 \in L^2(D, d\mathbf{x})$. We prove the following result.

PROPOSITION 7. *Assume $\psi_0 \in L^2(D, d\mathbf{x})$; then for $t\in[0,T]$,*

$$\lim_{N\to\infty}\lim_{M\to\infty} \|\, \psi(t) - \psi_{N,M}(t)\,\|^2 = 0. \tag{22}$$

Proof. Because we already proved that for each $N \ge 0$

$$\mathbb{E}\{\|\,\psi^N(t)\,\|^2\} \le \frac{1}{N!}\left(\frac{\sigma^2 t}{4}\right)^N e^{-\frac{\sigma^2}{4}t}\,\|\,\psi_0\,\|^2$$

for any $\varepsilon > 0$ there exists an N_0 such that for all $t \in [0,T]$ the following inequality holds:

$$\mathbb{E}\left\{\left\|\sum_{N>N_0}\psi^N(t)\right\|^2\right\} \le \varepsilon.$$

With this notation at hand,

$$\mathbb{E}\{\|\,(U-U_M)(t-s)\psi^N(s)\,\|^2\} = e^{-\sigma^2(t-s)/4}\sum_{(m,n)\in I_M^c}\mathbb{E}\{|\langle\psi^N(s), B_{m,n}\rangle|^2\},$$

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which converges toward 0 as M increases to ∞. According to Lemma 10 in the second part of the appendix,

$$\sum_{(m,n)\in I_M^c} \mathbb{E}[|\langle \psi^N(s), B_{m,n}\rangle|^2]$$

converges to 0 uniformly for all $N = 0, \dots, N_0$ and $0 \le s \le t \le T$. We have

$$\psi_M^N(t) - \psi^N(t) = -\frac{i}{2}\int_0^t U_M(t-s)\psi_M^{N-1}(s)W(ds) + \frac{i}{2}\int_0^t U(t-s)\psi^{N-1}(s)W(ds)$$
$$= \frac{i}{2}\left[\int_0^t (U-U_M)(t-s)\psi^{N-1}(s)W(ds) - \int_0^t U_M(t-s)[(\psi_M^{N-1}-\psi^{N-1})(s)]W(ds)\right].$$

Using the Hilbert–Schimdt norm from H to $L^2(D, d\mathbf{x})$, we obtain

$$4\mathbb{E}\{\|\psi_M^N(t) - \psi^N(t)\|^2\} = 2l\sum_{m,n}\alpha_{m,n}\int_0^t \mathbb{E}\{\|(U-U_M)(t-s)\psi^{N-1}(s)B_{-m,-n}\|^2\}ds$$
$$+\ 2l\sum_{m,n}\alpha_{m,n}\int_0^t \mathbb{E}\{\|U_M(t-s)(\psi_M^{N-1}-\psi^{N-1})(s)B_{-m,-n}\|^2\}ds.$$

The second summation on the right-hand side satisfies the inequality

$$2l\sum_{m,n}\alpha_{m,n}\int_0^t \mathbb{E}\{\|U_M(t-s)(\psi_M^{N-1}-\psi^{N-1})(s)B_{-m,-n}\|^2\}ds$$
$$\le 2l\sum_{m,n}\alpha_{m,n}\int_0^t \mathbb{E}\{\|U(t-s)(\psi_M^{N-1}-\psi^{N-1})(s)B_{-m,-n}\|^2\}ds$$
$$= \sigma^2\int_0^t \mathbb{E}\{e^{-\sigma^2(t-s)/4}\|\psi_M^{N-1}(s)-\psi^{N-1}(s)\|^2\}ds.$$

Hence we need only to estimate the first summation:

$$2l\sum_{m,n}\alpha_{m,n}\int_0^t \mathbb{E}\{\|(U-U_M)(t-s)\psi^{N-1}(s)B_{-m,-n}\|^2\}ds$$
$$= \frac{1}{2l}\sum_{m,n}\alpha_{m,n}\int_0^t \mathbb{E}\left\{\sum_{m',n'\in I_M^c} e^{-\frac{\sigma^2}{4}(t-s)}|\langle\psi^{N-1}(s), B_{m+m',n+n'}\rangle|^2\right\}ds$$
$$= \left(\sum_{m,n\in I_{M'}^c} + \sum_{m,n\in I_{M'}}\right)\frac{\alpha_{m,n}}{2l}\int_0^t \mathbb{E}\left\{\sum_{m',n'\in I_M^c} e^{-\frac{\sigma^2}{4}(t-s)}|\langle\psi^{N-1}(s), B_{m+m',n+n'}\rangle|^2\right\}ds$$
$$\le \sum_{m,n\in I_{M'}^c}\frac{\alpha_{m,n}}{2l}\int_0^t \mathbb{E}\{e^{-\frac{\sigma^2}{4}(t-s)}\|\psi^{N-1}(s)\|^2\}ds$$
$$+\sum_{m,n\in I_{M'}}\frac{\alpha_{m,n}}{2l}\int_0^t e^{-\frac{\sigma^2}{4}(t-s)}\sum_{m',n'\in I_M^c}\mathbb{E}\{|\langle\psi^{N-1}(s), B_{m+m',n+n'}\rangle|^2\}ds.$$

Because $\sum_{m,n}\alpha_{m,n} = 2l\sigma^2$, we can find a positive integer M' such that

$$\sum_{m,n\in I_{M'}^c}\alpha_{m,n} \le \varepsilon 2l\sigma^2.$$

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Also, from the uniform convergence of the series

$$\sum_{m,n} \mathbb{E}\{|\langle \psi^N(t), B_{m,n}\rangle|^2\}, \qquad t \in [0,T],\ N = 0,\ldots,N_0,$$

there exists an $M'' > 0$ such that

$$\sum_{m,n \in I^c_{M''}} \mathbb{E}\{|\langle \psi^N(t), B_{m,n}\rangle|^2\} \le \varepsilon, \qquad t \in [0,T],\ N = 0,\ldots,N_0.$$

Let us define M as $M = M' + M''$ so that for all $m, n \in I_{M'}$ and $m', n' \in I^c_M$, $|m + m'| \ge |m| - |m'| \ge M''$ or $|n + n'| \ge |n| - |n'| \ge M''$. Therefore

$$\begin{aligned}
&2l \sum_{m,n} \alpha_{m,n} \int_0^t \mathbb{E}\{\|(U - U_M)(t-s)\psi^{N-1}(s)B_{-m,-n}\|^2\} ds \\
&\quad \le \varepsilon\sigma^2 \int_0^t \mathbb{E}\{e^{-\frac{\sigma^2}{4}(t-s)} \|\psi^{N-1}(s)\|^2\} ds + \sum_{m,n \in I_{M'}} \frac{\alpha_{m,n}}{2l} \int_0^t \varepsilon dt \\
&\quad \le \left(\varepsilon \frac{1}{N!} \left(\frac{\sigma^2}{4}\right)^{N-1} t^N e^{-\sigma^2 t/4} \|\psi_0\|^2 + \varepsilon t\right)\sigma^2.
\end{aligned}$$

For $N = 1, \ldots, N_0$ and $t \in [0,T]$, we have the inequality

$$\begin{aligned}
&\mathbb{E}\ \{\|\psi^N_M(t) - \psi^N(t)\|^2\} \\
&\le \varepsilon \frac{1}{N!}\left(\frac{\sigma^2 t}{4}\right)^N e^{-\sigma^2 t/4} \|\psi_0\|^2 + \varepsilon\frac{\sigma^2 t}{4} + \frac{\sigma^2}{4}\int_0^t \mathbb{E}\{e^{-\sigma^2(t-s)/4} \|\psi^{N-1}_M(s) - \psi^{N-1}(s)\|^2\} ds \\
&\le \varepsilon \frac{1}{(N-1)!}\left(\frac{\sigma^2 t}{4}\right)^N e^{-\sigma^2 t/4} \|\psi_0\|^2 + \varepsilon \sum_{j=1}^N \frac{1}{j!}\left(\frac{\sigma^2 t}{4}\right)^j \\
&\quad + \left(\frac{\sigma^2}{4}\right)^N \int_0^t ds_N \int_0^{s_N} ds_{N-1} \cdots \int_0^{s_2} ds_1 e^{-\sigma^2(t-s_1)/4} \|\psi^0_M(s_1) - \psi^0(s_1)\|^2 \\
&\le \varepsilon \frac{\sigma^2 t}{4} \|\psi_0\|^2 + \varepsilon e^{\sigma^2 t/4} \\
&\quad + \left(\frac{\sigma^2}{4}\right)^N \int_0^t ds_N \int_0^{s_N} ds_{N-1} \cdots \int_0^{s_2} ds_1 e^{-\sigma^2(t-s_1)/4} \|\psi^0_M(s_1) - \psi^0(s_1)\|^2 .
\end{aligned}$$

Because

$$\|\psi^0_{M'''}(t) - \psi^0(t)\|^2 = e^{-\sigma^2 t/4} \sum_{(m,n) \in I^c_{M'''}} |\langle \psi_0, B_{m,n}\rangle|^2,$$

there exists $M^* > 0$ such that

$$\|\psi^0_{M^*}(t) - \psi^0(t)\|^2 \le e^{-\sigma^2 t/4}\varepsilon.$$

Let $M_\varepsilon = \max(M, M^*)$; we have

$$\begin{aligned}
\mathbb{E}\{\|\psi^N_{M_\varepsilon}(t) - \psi^N(t)\|^2\} &\le \varepsilon \frac{\sigma^2 t}{4} \|\psi_0\|^2 + \varepsilon e^{\sigma^2 t/4} + \varepsilon \frac{1}{N!}\left(\frac{\sigma^2 t}{4}\right)^N e^{-\sigma^2 t/4} \\
&\le \varepsilon \frac{\sigma^2 t}{4} \|\psi_0\|^2 + \varepsilon e^{\sigma^2 t/4} + \varepsilon, \qquad N = 1,\ldots,N_0,\ t \in [0,T].
\end{aligned}$$

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Therefore we can use

$$\psi_{N,M_{\varepsilon/N_0}}(t) = \sum_{N=0}^{N_0} \psi^N_{M_{\varepsilon/N_0}}(t)$$

to approximate $\psi(t)$ on the interval $t \in [0,T]$. $\square$

In fact, the above derivation also proves that if we assume that ψ_0 and $\partial\psi_0/\partial\gamma$ both belong to $L^2(D, d\mathbf{x})$, then we can simultaneously approximate $\psi(t)$ and $\partial\psi_0/\partial\gamma$.

5.2. Numerical scheme for the Green's function. As in the case of the existence of a mild solution, the analysis of the Green's function $G(t, \mathbf{y}; \mathbf{x})$ requires a special treatment. We control the approximations in the space H_β and its dual H^*_β introduced in subsection 2.1 associated with a sequence $\{\beta_{mn}\}_{m,n}$ satisfying the assumption (10).

PROPOSITION 8. (1) *The mild solution $\psi(t)$ belongs to H_β whenever $\psi_0 \in H_\beta$. Moreover, it is Malliavin differentiable, and its Malliavin derivative $D_s\psi(t)$ is a Hilbert–Schmidt operator from H into H^*_β.*

(2) *There exists a numerical scheme for the mild solution in H_β such that*

$$\lim_{N\to\infty} \lim_{M\to\infty} \| \psi(t) - \psi_{N,M}(t) \|^2_{H_\beta} = 0. \tag{23}$$

Recall that H_β contains the Dirac functions. Hence, the numerical scheme described above will be applicable to the Green's function. Also, note that the above error does not include the component due to the discretization in time.

Proof. We use the short notation $\|\cdot\|_\beta$ for $\|\cdot\|_{H_\beta}$. For $N = 1, \dots, N_0$,

$$\begin{aligned} 4\mathbb{E}\{\| \psi^N_M(t) - \psi^N(t) \|^2_\beta\} &= 2l \sum_{m,n} \alpha_{m,n} \int_0^t \mathbb{E}\{\| (U - U_M)(t-s)\psi^{N-1}(s)B_{-m,-n} \|^2_\beta\} ds \\ &\quad + 2l \sum_{m,n} \alpha_{m,n} \int_0^t \mathbb{E}\{\| U_M(t-s)(\psi^{N-1}_M - \psi^{N-1})(s)B_{-m,-n} \|^2_\beta\} ds. \end{aligned}$$

By choosing $M > 0$ such that

$$\sum_{(m',n')\in I^c_M} \beta_{m',n'} \le \varepsilon,$$

we have

$$\begin{aligned} &2l\mathbb{E}\{\| (U - U_M)(t-s)\psi^{N-1}(s)B_{-m,-n} \|^2_\beta\} \\ &\quad \le 2l \sum_{m',n'} \beta_{m',n'} \mathbb{E}[|\langle (U - U_M)(t-s)\psi^{N-1}(s)B_{-m,-n}, B_{m',n'}\rangle|^2] \\ &\quad = \sum_{m',n'\in I^c_M} \frac{\beta_{m',n'}}{2l} e^{-\sigma^2(t-s)/4} \mathbb{E}\{|\langle \psi^{N-1}(s), B_{m+m',n+n'}\rangle|^2\} \\ &\quad \le \left(\sum_{m',n'\in I^c_M} \frac{\beta_{m',n'}}{2l} \right) e^{-\sigma^2(t-s)/4} \sup_{m,n} \mathbb{E}\{|\langle \psi^{N-1}(s), B_{m,n}\rangle|^2\} \\ &\quad \le \left(\sum_{m',n'\in I^c_M} \frac{\beta_{m',n'}}{2l} \right) \frac{1}{(N-1)!} \left(\frac{\sigma^2 s}{4} \right)^{N-1} e^{-\sigma^2 t/4} \sup_{m,n} |\langle \psi_0, B_{m,n}\rangle|^2 \\ &\quad \le \frac{\varepsilon}{2l} \frac{1}{(N-1)!} \left(\frac{\sigma^2 s}{4} \right)^{N-1} e^{-\sigma^2 t/4} \sup_{m,n} |\langle \psi_0, B_{m,n}\rangle|^2. \end{aligned}$$

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Meanwhile,

$$
\begin{aligned}
2l\sum_{m,n}&\alpha_{m,n}\mathbb{E}[\| U_M(t-s)(\psi_M^{N-1}-\psi^{N-1})(s)B_{-m,-n}\|_\beta^2] \\
&=\sum_{m,n}\frac{\alpha_{m,n}}{2l}\sum_{m',n'\in I_M}\beta_{m',n'}e^{-\frac{\sigma^2}{4}(t-s)}\mathbb{E}\{|\langle\psi_M^{N-1}(s)-\psi^{N-1}(s),B_{m+m',n+n'}\rangle|^2\} \\
&\le\sum_{m,n}\frac{\alpha_{m,n}}{2l}\sum_{m',n'}\beta_{m',n'}e^{-\sigma^2(t-s)/4}\mathbb{E}\{|\langle\psi_M^{N-1}(s)-\psi^{N-1}(s),B_{m+m',n+n'}\rangle|^2\} \\
&=\frac{1}{2l}\sum_{m,n}\sum_{m',n'}\alpha_{m-m',n-n'}\beta_{m',n'}e^{-\sigma^2(t-s)/4}\mathbb{E}\{|\langle\psi_M^{N-1}(s)-\psi^{N-1}(s),B_{m,n}\rangle|^2\} \\
&\le\frac{C'}{2l}\sum_{m,n}\beta_{m,n}e^{-\sigma^2(t-s)/4}\mathbb{E}\{|\langle\psi_M^{N-1}(s)-\psi^{N-1}(s),B_{m,n}\rangle|^2\} \\
&=C\sigma^2e^{-\sigma^2(t-s)/4}\mathbb{E}\{\|\psi_M^{N-1}(s)-\psi^{N-1}(s)\|_\beta^2\}.
\end{aligned}
$$

Therefore, for $N=1,\dots,N_0$ and $t\in[0,T]$,

$$
\begin{aligned}
&\mathbb{E}\{\|\psi_M^N(t)-\psi^N(t)\|_\beta^2\} \\
&\le\varepsilon\frac{1}{N!}\left(\frac{\sigma^2t}{4}\right)^N e^{-\sigma^2t/4}\sup_{m,n}|\langle\psi_0,B_{m,n}\rangle|^2 \\
&\quad +C\frac{\sigma^2}{4}\int_0^t e^{-\sigma^2(t-s)/4}\mathbb{E}\{\|\psi_M^{N-1}(s)-\psi^{N-1}(s)\|_\beta^2\}ds \\
&\le(\max(1,C))^N\varepsilon\frac{1}{(N-1)!}\left(\frac{\sigma^2t}{4}\right)^N e^{-\sigma^2t/4}\sup_{m,n}|\langle\psi_0,B_{m,n}\rangle|^2 \\
&\quad +\left(\frac{\sigma^2C}{4}\right)^N\int_0^t ds_N\int_0^{s_N}ds_{N-1}\cdots\int_0^{s_2}ds_1e^{-\sigma^2(t-s_1)/4}\,\|\psi_M^0(s)-\psi^0(s)\|_\beta^2\,.
\end{aligned}
$$

Because

$$
\begin{aligned}
\|\psi_M^0(t)-\psi^0(t)\|_\beta^2&=e^{-\sigma^2t/4}\sum_{m,n\in I_M^c}\beta_{m,n}|\langle\psi^0,B_{m,n}rangle|^2 \\
&\le\varepsilon e^{-\sigma^2t/4}\sup_{m,n}|\langle\psi^0,B_{m,n}\rangle|^2,
\end{aligned}
$$

we get

$$
\begin{aligned}
&\mathbb{E}\{\|\psi_M^N(t)-\psi^N(t)\|_\beta^2\} \\
&\quad\le\varepsilon\frac{(\max(1,C))^N}{(N-1)!}\left(\frac{\sigma^2t}{4}\right)^N e^{-\sigma^2t/4}\sup_{m,n}|\langle\psi_0,B_{m,n}\rangle|^2+\varepsilon\frac{1}{N!}\left(\frac{\sigma^2Ct}{4}\right)^N e^{-\sigma^2t/4} \\
&\quad\le\varepsilon\max(1,C)\frac{\sigma^2t}{4}\sup_{m,n}|\langle\psi_0,B_{m,n}\rangle|^2e^{\sigma^2(max(1,C)-1)t/4}+\varepsilon e^{\sigma^2(C-1)t/4}.
\end{aligned}
$$

Replacing ε by ε/N_0, we can get the numerical scheme to converge in H_β. ∎

6. Numerical results. Before actually computing the sensitivities for the 3D problem, we tested our numerical scheme on the 2D case already studied numerically in [1] and [16].

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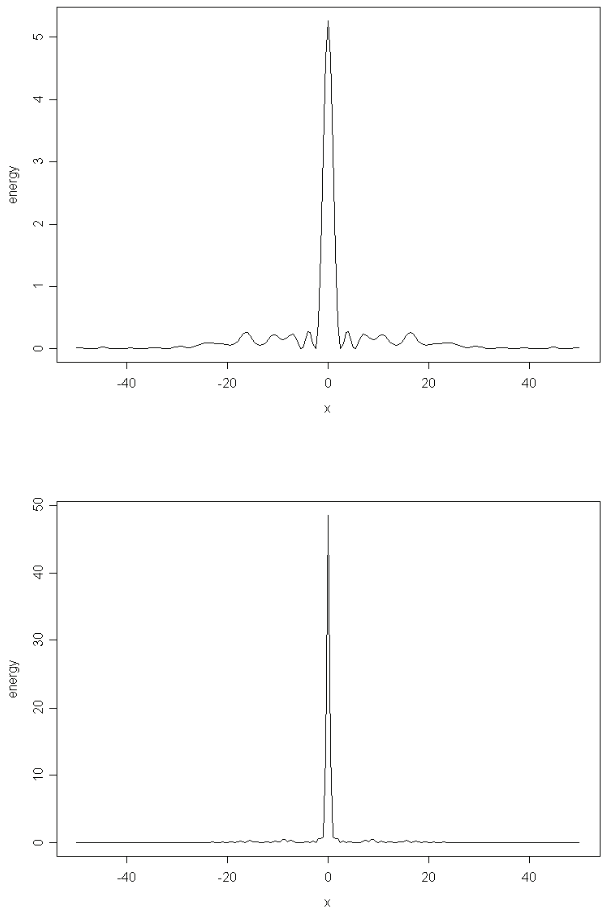

FIG. 2. *Energy of the time-reversed pulse on the line* $(x, z = 0)$*; the noise variance is* $\sigma^2 = 0.0367957$ *with nine terms in the decomposition of the Wiener field, and propagation distance in the* z*-direction is* $600m$ *with a step size of* $0.6m$*. For the Fourier transform, we used* 32 *wave numbers* k *centered at* 3 *with a step size* 0.15*. The SPDE is solved using a chaos expansion with* $N = 20$ *terms and* $M = 39$ *(top) and* $M = 119$ *(bottom) terms in the Fourier basis.*

6.1. The 2D case. In this case $D = [-l, +l]$, and we take $l = 50$m. Figure 2 shows the energy of the time-reversed pulse on the line $z = 0$. The SPDE was solved with a chaos expansion with $N = 20$ terms and with $M = 39$ (top) and $M = 119$ (bottom) terms in the Fourier basis. In both cases, the variance of the white noise was chosen to be $\sigma^2 = 0.0367957$, the Wiener field being simulated from a Karhunen–Loeve expansion with nine terms. The propagation distance was $L = 600$m with a step size of 0.6m for the numerical computation of the integral. We used 32 wave numbers k centered at 3 and a step size of 0.15 to compute the integrals giving the Fourier transform. The corresponding frequencies are given by the wave numbers multiplied by the sound speed, which is assumed to be $c_0 = 1500\,$m/s. Hence the 32 frequencies ω range between 900 Hz and 8100 Hz. Since they are not present in the bottom panel, the explanation for the presence of lobes in the top panel can only be

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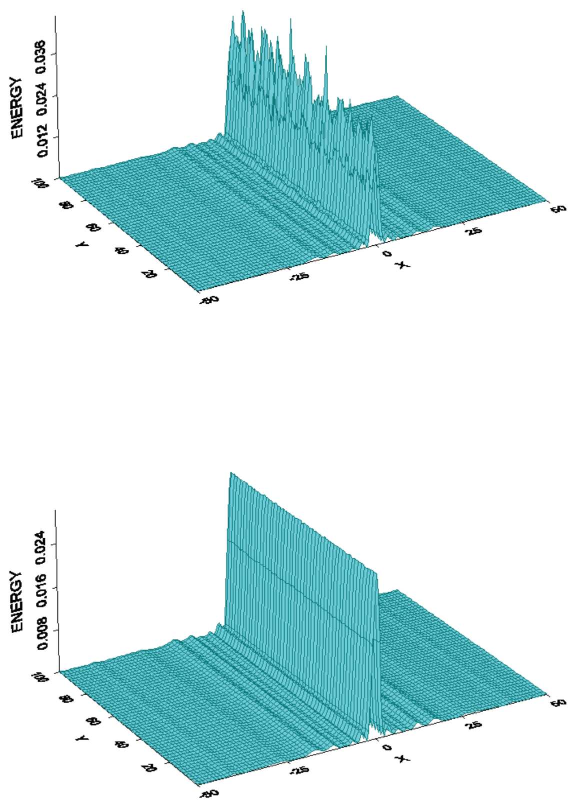

Fig. 3. *Illustration of the self-averaging property of the energy of the time-reversed pulse. Plots obtained for* 100 *different realizations of the white noise with one term (top) and nine terms (bottom) in the decomposition of the Wiener field, with all other parameters identical: the noise variance is* 0.0324*, and the propagation distance is* 1000 *m with step size* 1 *m. For the Fourier transform, we used* 32 *wave numbers centered at* 3 *with a step size* 0.15*. The SPDE is solved with a chaos expansion with* 20 *terms and* 39 *terms in the Fourier basis.*

the poor approximation resulting from the use of only 39 basis elements in the Fourier basis. Finally, the mirror region is $A = [-10, +10] \times \{L\}$.

One of the important properties reviewed in the introduction was the self-averaging property of the time-reversed field proven in [15]. We illustrate this result by plotting the energy of the time-reversed pulse (again on the line $z = 0$) for twenty different realizations of the white noise (all other parameters remaining unchanged). As we can see from Figure 3, these twenty plots look identical, confirming the self-averaging property.

Our results compare very favorably to the numerical results reported in [1] and [16]. Because it does not rely on finite difference or finite element methods, our numerical scheme is faster and more robust. However, because it is based on the

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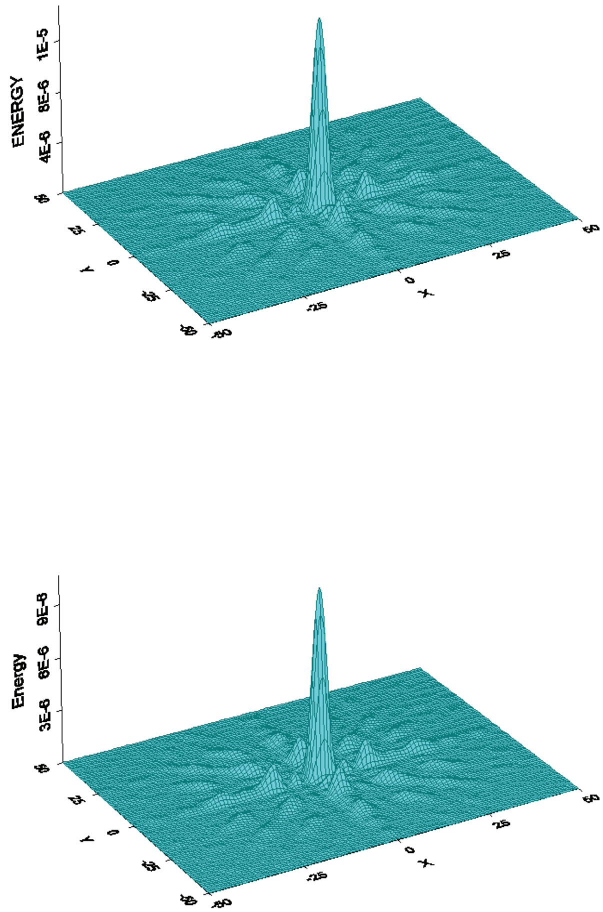

Fig. 4. *Surface plots of the time-reversed field without cutoff on the time-reversal mirror on the plane* $z = 0$ *(top) and with cut off* $s_0 = 0.0001$ *(bottom). Both computations use the same set of parameters given in the text. Even though the two plots are qualitatively similar, a careful look at the scale on the vertical axis shows the significant differences caused by the thresholding.*

Wiener chaos expansion, it is limited to the white noise limiting regime of the parabolic approximation.

6.2. The 3D case. The top panel of Figure 4 gives the surface plot of the time-reversed field in a 3D numerical experiment without cutoff, while the bottom panel of the same figure gives the surface plot of the time-reversed field in the same 3D numerical experiment with a cutoff on the time-reversal mirror($s_0 = 0.00001$). The plots are given over the region $[-50, 50] \times [-50, 50] \times \{0\}$. As before, the propagation distance is $L = 1000$ m, and we used a step size of 1 m in the numerical computation of

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the inverse Fourier integral. The mirror region is now $A = [-10, 10] \times [-10, 10] \times \{L\}$. The noise variance is 0.0324 with 361 terms in the decomposition of the Wiener field. We used 10 terms in the chaos expansion and 441 terms in the (x_1, x_2) Fourier basis. We used 32 wave numbers centered at 3 with a step size of 0.15. The sound speed is 1500 m/s, so the frequencies are between 900 Hz and 8100 Hz. With the same settings, we compute the sensitivity of the time-reversed wave with respect to the shape parameter γ. For the purpose of the numerical experiment, we chose the amplitude $|\psi_0|$ of the initial pulse ψ_0 to be a Gaussian with width (standard deviation) γ. We computed approximations of the sensitivities based on Malliavin derivatives for $\gamma = 0.1$. The results are presented in Figure 5.

Appendix. This appendix contains the proofs of two technical estimates used in the paper. We give complete proofs because they play a crucial role, but because of their technical nature, we postponed their presentation to an appendix.

A.1. Existence of the sequence $\{\beta_{mn}\}_{m,n}$. The series $\{\beta_{m,n}\}_{m,n}$ plays an important role in the derivation of the numerical scheme for the Green's function. The following lemma shows the existence of sequences $\{\beta_{m,n}\}_{m,n}$ satisfying condition (10) whenever the spectral sequence $\{\alpha_{m,n}\}_{m,n}$ satisfies a mild decay condition.

LEMMA 9. *Suppose that the spectral density $\{\alpha_{m,n}\}_{m,n}$ satisfies the condition*

$$\alpha_{m,n} \leq \frac{C}{(1+m^2+n^2)^{2k}}, \qquad m,n \in \mathbb{Z}, \tag{24}$$

for some positive constant C and a constant $k > 1$. Then there exists a sequence $\{\beta_{m,n}\}_{m,n}$ such that for some $C' > 0$,

$$\sum_{m',n'} \alpha_{m-m',n-n'} \beta_{m',n'} \leq C' \beta_{m,n}, \qquad m,n \in \mathbb{Z}.$$

Proof. Let $\beta_{m,n} = (1+m^2+n^2)^{-k}$. We have

$$\begin{aligned}
&\sum_{m',n'} \alpha_{m-m',n-n'} \frac{\beta_{m',n'}}{\beta_{m,n}} \\
&\leq \sum_{m',n'} \frac{C}{(1+(m-m')^2+(n-n')^2)^k} \frac{(1+m^2+n^2)^k}{(1+(m-m')^2+(n-n')^2)^k (1+m'^2+n'^2)^k} \\
&\leq \sum_{m',n'} \frac{C}{(1+(m-m')^2+(n-n')^2)^k} \frac{(1+m^2+n^2)^k}{(1+(m-m')^2+m'^2+(n-n')^2+n'^2)^k} \\
&\leq \sum_{m',n'} \frac{C}{(1+(m-m')^2+(n-n')^2)^k} \frac{(1+m^2+n^2)^k}{(1+(\frac{m}{2})^2+(\frac{n}{2})^2)^k} \\
&= \sum_{m',n'} \frac{C}{(1+(m-m')^2+(n-n')^2)^k} \frac{4^k(1+m^2+n^2)^k}{(4+m^2+n^2)^k} \\
&= 4^k \sum_{m',n'} \frac{C}{(1+(m-m')^2+(n-n')^2)^k} \\
&= 4^k \sum_{m,n} \frac{C}{(1+m^2+n^2)^k},
\end{aligned}$$

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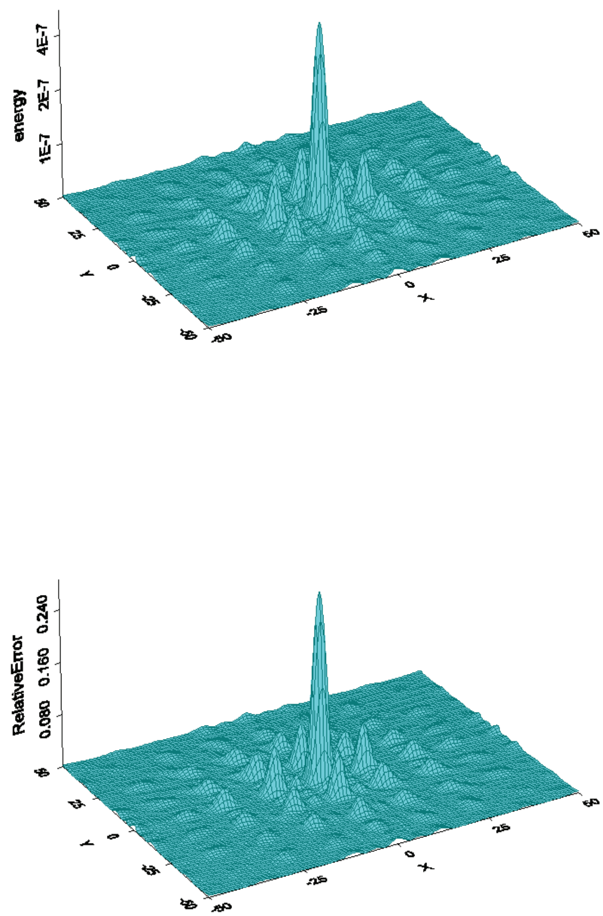

FIG. 5. *Surface plots of the Malliavin derivative and its relative errors on the plane $z = 0$ for $\gamma = 0.1$. We use 10 simulations to compute the relative errors. The parameters are the same as before. Naturally, errors are greater when the energy is larger, so the general shapes of the two plots are the same. However, there are significant differences in the vertical scales.*

which is a finite constant, say C', because $k > 1$. So we have proven the desired inequality

$$\sum_{m',n'} \alpha_{m-m',n-n'} \beta_{m',n'} \leq C' \beta_{m,n}$$

for all m and n. $\square$

A.2. Uniform continuity of the chaos expansion approximations. Here we give the lemma which is important to the numerical scheme for the functions in $L^2(D, d\mathbf{x})$.

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LEMMA 10. *Suppose* $\psi_0 \in L^2(F, d\mathbf{x})$. *For each* $N \geq 0$ *and* $0 \leq s \leq t \leq T$, *we have*

$$\lim_{t-s\to 0} \mathbb{E}[\|\,\psi^N(t) - \psi^N(s)\,\|^2] = 0,$$

and

$$\sum_{m,n\in I_M^c} \mathbb{E}[|\langle \psi^N(t), B_{m,n}\rangle|^2]$$

converges to 0 *uniformly.*

Proof. We prove the lemma by induction.

Step 1. When $n = 0$, $\psi^0(t) - \psi^0(s) = [U(t) - U(s)]\psi_0$ and

$$\begin{aligned}
&\mathbb{E}[\|\,\psi^0(t) - \psi^0(s)\,\|^2] \\
&= \mathbb{E}[\|\,U(s)(U(t-s) - 1)\psi_0\,\|^2] \\
&= e^{-\frac{\sigma^2 s}{4}} \sum_{m,n} \left[e^{-\frac{\sigma^2(t-s)}{4}} + 1 - 2e^{-\frac{\sigma^2(t-s)}{8}} \cos\frac{\lambda_{m,n}(t-s)}{2k} \right] |\langle \psi_0, B_{m,n}\rangle|^2.
\end{aligned}$$

For all $\varepsilon > 0$, there exists an $M > 0$ such that

$$\sum_{(m,n)\in I_M^c} |\langle \psi_0, B_{m,n}\rangle|^2 < \varepsilon;$$

therefore

$$\begin{aligned}
&\mathbb{E}[\|\,\psi^0(t) - \psi^0(s)\,\|^2] \\
&\leq e^{-\frac{\sigma^2 s}{4}} \sum_{(m,n)\in I_M} \left[e^{-\frac{\sigma^2(t-s)}{4}} + 1 - 2e^{-\frac{\sigma^2(t-s)}{8}} \cos\frac{\lambda_{m,n}(t-s)}{2k} \right] \|\,\psi_0\,\|^2 \\
&\quad + e^{-\frac{\sigma^2 s}{4}} \sum_{(m,n)\in I_M^c} \left(e^{-\frac{\sigma^2(t-s)}{8}} + 1 \right)^2 |\langle \psi_0, B_{m,n}\rangle|^2 \\
&\leq \sum_{(m,n)\in I_M} \left[e^{-\frac{\sigma^2(t-s)}{4}} + 1 - 2e^{-\frac{\sigma^2(t-s)}{8}} \cos\frac{\lambda_{m,n}(t-s)}{2k} \right] \|\,\psi_0\,\|^2 + 4\varepsilon.
\end{aligned}$$

Because the indices in I_M are finitely many,

$$\sum_{(m,n)\in I_M} \left[e^{-\frac{\sigma^2(t-s)}{4}} + 1 - 2e^{-\frac{\sigma^2(t-s)}{8}} \cos\frac{\lambda_{m,n}(t-s)}{2k} \right]$$

converges to zero uniformly while $|t - s| \to 0$, and we readily conclude that

$$\lim_{t-s\to 0} \mathbb{E}[\|\,\psi^0(t) - \psi^0(s)\,\|^2] = 0.$$

Notice that

$$\begin{aligned}
\sum_{m,n\in I_M^c} \mathbb{E}[|\langle \psi^0(t), B_{m,n}\rangle|^2] = 2 &\sum_{m,n\in I_M^c} \mathbb{E}[|\langle \psi^0(t) - \psi^0(s), B_{m,n}\rangle|^2] \\
&+ 2 \sum_{m,n\in I_M^c} \mathbb{E}[|\langle \psi^0(s), B_{m,n}\rangle|^2],
\end{aligned}$$

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and thus it is obvious that

$$\sum_{m,n \in I_M^c} \mathbb{E}[|\langle \psi^0(t), B_{m,n}\rangle|^2]$$

converges to 0 uniformly.

Step 2. Suppose that the statement is true when $N = N_0$. For $N = N_0 + 1$,

$$\begin{aligned}
&\psi^{N_0+1}(t) - \psi^{N_0+1}(s) \\
&= -\frac{i}{2}\int_0^t U(t-v)\psi^{N_0}(v)W(dv) + \frac{i}{2}\int_0^s U(s-v)\psi^{N_0}(v)W(dv) \\
&= -\frac{i}{2}\int_s^t U(t-v)\psi^{N_0}(v)W(dv) + \frac{i}{2}[U(t-s)-1]\int_0^s U(s-v)\psi^{N_0}(v)W(dv) \\
&= -\frac{i}{2}\int_s^t U(t-v)\psi^{N_0}(v)W(dv) - [U(t-s)-1]\psi^{N_0+1}(s).
\end{aligned}$$

It is easy to show that the first term goes to 0 uniformly in $L^2(\Omega \times D)$ as $t - s \to 0$. So we focus on the second term:

$$\begin{aligned}
&\mathbb{E}[\|\,[U(t-s)-1]\psi^{N_0+1}(s)\,\|^2] \\
&= \sum_{m,n}\left[e^{-\frac{\sigma^2(t-s)}{4}} + 1 - 2e^{-\frac{\sigma^2(t-s)}{8}}\cos\frac{\lambda_{m,n}(t-s)}{2k}\right]\mathbb{E}[\left|\langle \psi^{N_0+1}(s), B_{m,n}\rangle\right|^2],
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}[\left|\langle \psi^{N_0+1}(s), B_{m,n}\rangle\right|^2] \\
&= \frac{1}{8l}\int_0^s e^{-\frac{\sigma^2(s-v)}{4}}\sum_{m',n'}\alpha_{m',n'}\mathbb{E}\left\{\left|\int_{\mathbf{x}\in D}\psi^{N_0}(v,\mathbf{x})\overline{B_{m+m',n+n'}(\mathbf{x})}d\mathbf{x}\right|^2\right\}dv.
\end{aligned}$$

For all $\varepsilon > 0$, there exist $M', M'' > 0$ such that

$$\sum_{(m,n)\in I_{M'}^c}\frac{\alpha_{m,n}}{8l} < \varepsilon$$

and

$$\forall t \in [0,T], \sum_{(m,n)\in I_{M''}^c}\mathbb{E}[|\langle \psi^{N_0}(t), B_{m,n}\rangle|^2] < \varepsilon.$$

Let $M = M' + M''$; then

$$\begin{aligned}
&\mathbb{E}[\|\,[U(t-s)-1]\psi^{N_0+1}(s)\,\|^2] \\
&= \left[\sum_{(m,n)\in I_M} + \sum_{(m,n)\in I_M^c}\right]\left(e^{-\frac{\sigma^2(t-s)}{4}} + 1 - 2e^{-\frac{\sigma^2(t-s)}{8}}\cos\frac{\lambda_{m,n}(t-s)}{2k}\right) \\
&\quad\cdot\mathbb{E}[|\langle \psi^{N_0+1}(s), B_{m,n}\rangle|^2] \\
&\le \sum_{(m,n)\in I_M}\left(e^{-\frac{\sigma^2(t-s)}{4}} + 1 - 2e^{-\frac{\sigma^2(t-s)}{8}}\cos\frac{\lambda_{m,n}(t-s)}{2k}\right)\mathbb{E}[\|\,\psi^{N_0+1}(s)\,\|^2] \\
&\quad + \sum_{(m,n)\in I_M^c}\sum_{m',n'}\frac{1}{2l}\int_0^s e^{-\frac{\sigma^2(s-v)}{4}}\sum_{m',n'}\alpha_{m',n'}\mathbb{E}\left\{\left|\int_{\mathbf{x}\in D}\psi^{N_0}(v,\mathbf{x})\overline{B_{m+m',n+n'}(\mathbf{x})}d\mathbf{x}\right|^2\right\}dv.
\end{aligned}$$

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We have the inequality for the second term

$$\begin{aligned}
&\sum_{(m,n)\in I_M^c}\sum_{m',n'}\frac{1}{2l}\int_0^s e^{-\frac{\sigma^2(s-v)}{4}}\sum_{m',n'}\alpha_{m',n'}\mathbb{E}\left\{\left|\int_{\mathbf{x}\in D}\psi^{N_0}(v,\mathbf{x})\overline{B_{m+m',n+n'}(\mathbf{x})}d\mathbf{x}\right|^2\right\}dv\\
=\quad &\sum_{(m,n)\in I_M^c}\sum_{(m',n')\in I_{M'}^c}\frac{\alpha_{m',n'}}{2l}\int_0^s e^{-\frac{\sigma^2(s-v)}{4}}\mathbb{E}[|\langle\psi^{N_0}(v),B_{m+m',n+n'}\rangle|]dv\\
&+\sum_{(m',n')\in I_{M'}}\frac{1}{2l}\int_0^s e^{-\frac{\sigma^2(s-v)}{4}}\alpha_{m',n'}\sum_{(m,n)\in I_M^c}\mathbb{E}[|\langle\psi^{N_0}(v),B_{m+m',n+n'}\rangle|]dv\\
<\quad &4\varepsilon\int_0^s e^{-\frac{\sigma^2(s-v)}{4}}\mathbb{E}[\|\psi^{N_0}(v)\|^2]dv+\varepsilon\sigma^2\int_0^s e^{-\frac{\sigma^2(s-v)}{4}}dv\\
\leq\quad &\frac{16\varepsilon}{\sigma^2}\frac{1}{(N_0+1)!}\left(\frac{\sigma^2 s}{4}\right)^{N_0+1}e^{-\sigma^2 s/4}\|\psi_0\|^2+4\varepsilon\\
\leq\quad &4\varepsilon\left(\frac{4}{\sigma^2}\|\psi_0\|^2+1\right).
\end{aligned}$$

Because

$$\mathbb{E}[\|\psi^{N_0+1}(s)\|^2]\leq\frac{1}{(N_0+1)!}\left(\frac{\sigma^2 s}{4}\right)^{N_0+1}e^{-\sigma^2 s/4}\|\psi_0\|^2\leq\|\psi_0\|^2,$$

the first term satisfies

$$\begin{aligned}
&\sum_{(m,n)\in I_M}\left(e^{-\frac{\sigma^2(t-s)}{4}}+1-2e^{-\frac{\sigma^2(t-s)}{8}}\cos\frac{\lambda_{m,n}(t-s)}{2k}\right)\mathbb{E}[\|\psi^{N_0+1}(s)\|^2]\\
\leq&\sum_{(m,n)\in I_M}\left(e^{-\frac{\sigma^2(t-s)}{4}}+1-2e^{-\frac{\sigma^2(t-s)}{8}}\cos\frac{\lambda_{m,n}(t-s)}{2k}\right)\|\psi_0\|^2.
\end{aligned}$$

I_M is a finite set, so there exists $\eta>0$ such that for all $0\leq s\leq t\leq T$, if $|t-s|<\eta$,

$$\sum_{(m,n)\in I_M}\left(e^{-\frac{\sigma^2(t-s)}{4}}+1-2e^{-\frac{\sigma^2(t-s)}{8}}\cos\frac{\lambda_{m,n}(t-s)}{2k}\right)\mathbb{E}[\|\psi^{N_0+1}(s)\|^2]<\varepsilon.$$

Therefore

$$\lim_{|t-s|\to 0,s,t\in[0,T]}\mathbb{E}[\|\psi^{N_0+1}(t)-\psi^{N_0+1}(s)\|^2]=0.$$

Furthermore, it is easy to show that

$$\sum_{m,n\in I_M^c}\mathbb{E}[|\langle\psi^{N_0+1}(t),B_{m,n}\rangle|^2]$$

converges to 0 uniformly as M increases to ∞. The statement holds for $N=N_0+1$. By induction, the statement is true for all $N\geq 0$. ◻

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REFERENCES

[1] G. Bal and L. Ryzhik, *Time reversal and refocusing in random media*, SIAM J. Appl. Math., 63 (2003), pp. 1475–1498.

[2] F. Bailly, J. F. Clouet, and J. P. Fouque, *Parabolic and Gaussian white noise approximation for wave propagation in random media*, SIAM J. Appl. Math., 56 (1996), pp. 1445–1470.

[3] F. Bailly and J. P. Fouque, *High Frequency Wave Propagation in Random Media*, unpublished.

[4] R. Carmona and M. Tehranchi, *Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective*, Springer-Verlag, Berlin, 2006.

[5] R. Carmona and M. Tehranchi, *A characterization of hedging portfolios for interest rate contingent claims*, Ann. Appl. Probab., 14 (2004), pp. 1267–1294.

[6] D. Dawson and H. Salehi, *Spatially homogeneous random evolutions*, J. Multivariate Anal., 10 (1980), pp. 141–180.

[7] D. Dawson and G. Papanicolaou, *A Random wave process*, Appl. Math. Optim., 12 (1994), pp. 97–114.

[8] M. Fehler, H. Sato, and L. J. Huang, *Envelope broadening of outgoing waves in 2D random media: A comparison between the Markov approximation and numerical simulations*, Bull. Seismol. Soc. Amer., 90 (2000), pp. 914–925.

[9] S. M. Flatte and M. D. Vera, *Comparison between ocean-acoustic fluctuations in parabolic-equation simulations and estimates from integral approximations*, J. Acoust. Soc. Amer., 114 (2003), pp. 697–706.

[10] E. Fournié, J. Lasry, J. Lebuchoux, and P. Lions, *Applications of Malliavin calculus to Monte Carlo methods in finance* II, Finance and Stochastics, 5 (2001), pp. 201–236.

[11] E. Fournié, J. Lasry, J. Lebuchoux, P. Lions, and N. Touzi, *Applications of Malliavin calculus to Monte Carlo methods in finance* I, Finance and Stochastics, 3 (1999), pp. 391–412.

[12] T. Y. Hou, W. Luo, B. Rozovskii, and H. M. Zhou, *Wiener chaos expansions and numerical solutions of randomly forced equations of fluid dynamics*. J. Comput. Phys., 216 (2006), pp. 687–706.

[13] R. Mikulevicius and B. L. Rozovskii, *Stochastic Navier–Stokes equations for turbulent flows*, SIAM J. Math. Anal., 35 (2004), pp. 1250–1310.

[14] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer-Verlag, Berlin, 1995.

[15] G. Pananicolaou, L. Ryzhik, and K. Sølna, *Statistical stability in time reversal*, SIAM J. Appl. Math., 64 (2004), pp. 1133–1155.

[16] G. Pananicolaou, L. Ryzhik, and K. Solna, *The parabolic wave approximation and time reversal*, Mat. Contemp., 23 (2002), pp. 139–160.

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