Optimal Execution Tracking a Benchmark

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Optimal Execution Market Set-Up

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Goal: sell $v > 0$ shares by time $T > 0$ (finite horizon)

- P_t **mid-price** (unaffected price),

$$P_t = P_0 + \int_0^t \sigma(u)dW_u, \qquad 0 \leq t \leq T,$$

- $V(t)$ volume traded in the market up to (and including) time t
- Market **VWAP**$= \frac{1}{V}\int_0^T P_t dV(t)$
- Fraction of shares still to be executed in the market

$$X(t) = \frac{V - V(t)}{V} = \frac{T - t}{T}$$

(deterministic $V(t)$ used to **change clock**). **Convenient simplification** !

Broker Problem

v_t volume executed by the broker up to time t

$$x_t = \frac{v - v_t}{v}$$

fraction of shares left to be executed by the broker at time t

$$x_t = 1 - \ell_t - m_t$$

Where

- ℓ_t **cumulative** volume executed through **limit orders**
- m_t **cumulative** volume executed through **market orders**
- Broker average liquidation price **vwap**$= \frac{1}{v} \int_0^T \left(P_t - \frac{S}{2}\right) dm_t + \left(P_t + \frac{S}{2}\right) d\ell_t$
- **Objective:** Minimize discrepancy between **vwap** and **VWAP**

Naive Model for the Dynamics of the Order Book

Controls of the broker:

- $(m_t)_{0\le t\le T}$ **non-decreasing** adapted process
- $(L_t)_{0\le t\le T}$ **predictable** process

$$\ell_t = \int_0^t \int_{[0,1]} y \wedge L_u \, \mu(du, dy) = \sum_{i=1}^{N_t} Y_i \wedge L_{\tau_i}$$

where

$$\mu(du, dy)$$

point measure (Poisson) compensator $\nu_t(du)\nu(t)dt$.

$$x_t = 1 - \int_0^t \int_{[0,1]} y \wedge L_u \, \mu(du, dy) - m_t = 1 - \sum_{i=1}^{N_t} Y_i \wedge L_{\tau_i} - m_t$$

So the dynamics of x_t are given by

$$dx_t = -\int_{[0,1]} y \wedge L_t \, \mu(dt, dy) - dm_t,$$

with initial condition $x_{0-} = 1$.

Optimization Problem

Goal of the broker

$$\sup_{(\underline{L},\underline{m})\in\mathcal{A}} \mathbb{E}\Big[U(\text{vwap} - \text{VWAP})\Big],$$

For the CARA exponential utility, approximately

$$\inf_{(\underline{L},\underline{m})\in\mathcal{A}} \mathbb{E}\Bigg[\exp\Bigg(-\gamma\bigg(\frac{S}{2} + \int_0^T [x_u^{L,m} - X(u)]dP_u - S\,dm_u\bigg)\Bigg)\Bigg],$$

We will work with a **Mean - Variance** criterion

$$\inf_{(\underline{L},\underline{m})\in\mathcal{A}} \mathbb{E}\Bigg[\int_0^T \gamma\frac{\sigma(u)^2}{2}[x_u^{L,m} - X(u)]^2 du + S\,m_T\Bigg],$$

- S spread
- $X(u) = (T-u)/T$ fraction of shares left to be executed in the market.

Stochastic Control Problem

Singular control problem of a pure jump process

Value function

$$J(t,x) = \inf_{(\underline{L},\underline{m}) \in \mathcal{A}(t,x)} J(t,x,\underline{L},\underline{m})$$

where

$$J(t,x,\underline{L},\underline{m}) = \mathbb{E}\left[\int_t^T \gamma \frac{\sigma(u)^2}{2} [x_u^{L,m} - X(u)]^2 du + S m_T \right].$$

$J(t,x)$ **is non-decreasing in** t **for** $x \in [0,1]$ **fixed.** $(\mathcal{A}(t_2,x) \subset \mathcal{A}(t_1,x)$ whenever $t_1 \leq t_2)$

Tough Luck: Problem is NOT Convex

The set $\mathcal{A}$ of admissible controls is not convex.

For any number $\ell \in (0,1)$, the two controls $(\underline{L}^1, \underline{m}^1)$ and $(\underline{L}^2, \underline{m}^2)$ by:

$$L_t^1 = \mathbf{1}_{\{t \le \tau_1\}} + \sum_{k=2}^{\infty} x_{\tau_{k-1}} \mathbf{1}_{\{\tau_{k-1} < t \le \tau_k\}}, \qquad \text{and} \qquad m_t^1 = x_{T-} \mathbf{1}_{\{T \le t\}},$$

and:

$$L_t^2 = \frac{\ell}{2} \mathbf{1}_{\{t \le \tau_1\}} + \sum_{k=2}^{\infty} x_{\tau_{k-1}} \mathbf{1}_{\{\tau_{k-1} < t \le \tau_k\}}, \qquad \text{and} \qquad m_t^2 = x_{T-} \mathbf{1}_{\{T \le t\}},$$

are admissible, but the pair $(\underline{L}, \underline{m})$ defined by

$$L_t = \frac{1}{2}(L_t^1 + L_t^2), \qquad \text{and} \qquad m_t = \frac{1}{2}(m_t^1 + m_t^2),$$

IS NOT

Closest Related Works

- Poisson random measure $\mu(dt, dy)$ for claim sizes Y_t
- **insurer** pays $Y_t \wedge \alpha_t$ up to a **retention level** α_t
- **re-insurer** covers the excess $(Y_t - \alpha_t)^+$

Wealth process of the Insurance Company

$$X_t = x + \int_0^t p(\alpha_s)ds - \int_0^t y \wedge \alpha_s \, \mu(ds, dy) - \int_0^t dD_s$$

- $p(\alpha)$ insurer net premium (after paying the reinsurance company)
- D_t cumulative dividends paid up to (and including) time t

$$\sup_{(\alpha_t)_t, (D_t)_t} \mathbb{E}\left[\int_0^\tau e^{-ru} dD_u\right]$$

- time of bankruptcy $\tau = \inf\{t \geq 0;\; X_t \leq 0\}$

Jeanblanc-Shyryaev (1995) optimal dividend distribution for Wiener process,
Asmunssen- Hjgaard-Taksar (1998) optimal dividend distribution for diffusion,
Mnif-Sulem (2005) prove existence and uniqueness of a viscosity solution,
Goreac (2008) multiple contracts

Similarities & Differences

Similarities

- $\alpha_t \leftrightarrow$ standing limit orders L_t
- $D_t \leftrightarrow$ cumulative market orders m_t

Differences

- We work in a **finite horizon** (PDEs instead of ODEs)
- We use a **Mean - Variance** criterion
- We exhibit a **classical** solution (as opposed to a viscosity solution)
- We derive a **system of ODEs** identifying
  - the value function
  - the optimal stratagy

Technical Assumptions

$\nu_t(dy)\nu(t)dt$ intensity of Poisson measure $\mu(dt, dy)$ with $\nu_t([0,1]) = 1$.

- $\int_0^T \sigma(t)^2 dt < \infty$
- $\sup_{0 \leq t \leq T} \nu(t) < \infty$
- $t \hookrightarrow \frac{\sigma(t)^2}{\nu(t)}(X(t) - x)$ is increasing for each $x \in [0,1]$
- $t \hookrightarrow \frac{1}{\nu(t)}\nu_t(\cdot)$ is decreasing (in the sense of *stochastic dominance*)

Hamilton-Jabobi-Bellman Equation (QVI)

$$\min\left[[A\phi](t,x), \partial_t\phi(t,x) + [B\phi](t,x)\right] = 0.$$

where

$$[A\phi](t,x) = S - \partial_x\phi(t,x)$$

and

$$[B\phi](t,x) = \gamma\frac{\sigma(t)^2}{2}[X(t)-x]^2 + \nu(t)\inf_{0\le L\le x}\int_{[0,1]}[\phi(t,x-y\wedge L)-\phi(t,x)]\nu_t(dy)$$

with terminal condition

$$\phi(T-,x) = Sx, \qquad \text{(notice that } \phi(T,x)=0\text{)}$$

and boundary condition:

$$\phi(t,0) = \int_t^T \frac{\gamma\sigma(u)^2}{2}X(u)du.$$

Classical Solution

Theorem

*The value function is the **unique** solution of*

$$-\dot{J}(t,x) = \min\Bigg[\inf_{0\le y\le x} -\dot{J}(t,x),$$

$$\gamma\frac{\sigma(t)^2}{2}[X(t)-x]^2 + \nu(t)\int_{[0,1]}[J(t,(x-y)\vee\tilde{L}(t,y)) - J(t,x)]\nu_t(dy)\Bigg]$$

with

$$J(t,0) = \gamma\int_0^t \frac{\sigma(u)^2}{2}X(u)^2du, \qquad \textit{and} \qquad J(T,x) = Sx$$

where

$$\tilde{L}(t,x) = \arg\min_{0\le y\le x} J(t,y)$$

- J is $C^{1,1}$
- $x \hookrightarrow J(t,x)$ convex for t fixed
- $t \hookrightarrow J(t,x)$ non-decreasing for x fixed
- $\partial_x \dot{J}(t,x) \geq 0$

Free Boundary (No-Trade Region)

$$[0,T]\times[0,1] = A\cup B\cup C$$

with

- $A = \{(t,x); \partial_x J(t,x) < 0\} = \{(t,x);\ 0\le t<\tau_\ell(x)\}$
- $B = \{(t,x); 0\le \partial_x J(t,x) \le S\} = \{(t,x);\ \tau_\ell(x)\le t\le \tau_m(x)\}$
- $C = \{(t,x); \partial_x J(t,x) = S\} = \{(t,x);\ \tau_m(x)\le t\}$

where

- $\tau_\ell(x) = \inf\{t>0;\ \partial_x J(t,x)\ge 0\}$
- $\tau_m(x) = \inf\{t>0;\ \partial_x J(t,x)\ge S\}$

$$\tau_\ell(x)\le T(1-x)\le \tau_m(x)$$

Optimal Trading Strategy

- If $t > \tau_m(x_t)$ i.e. $(t, x_t) \in C$ (never happens)
  - **place market orders**

  $\Delta m_t > 0$ (just enough to get into B)
- If $t = \tau_m(x_t)$ i.e. $(t, x_t) \in \partial C$
  - **place market orders at a rate** $dm_t = -\dot{\tau}_m(x_t)dt$

  (just enough so not to exit B)
- If $\tau_\ell(x_t) \leq t < \tau_m(x_t)$ i.e. $(t, x_t) \in B \cup \partial A$
  - **place** $L_t = x_t - \tilde{L}(t)$ **limit orders**

  (as much as possible without getting ahead too much)
- If $t < \tau_\ell(x_t)$ i.e. $(t, x_t) \in A$ (never happens)
  - **no trade**

Special Case I: Large Fill Distribution

$\nu_t(dy) = \delta_1(dy)$: the crossings, when they occur, fill all the requested limit orders.

Theorem

The value function solves

$$-\dot{J}(t,x) = \min\left[\inf_{0\le y\le x} -\dot{J}(t,x), \gamma\frac{\sigma(t)^2}{2}[X(t)-x]^2 + \nu(t)[J(t,\tilde{L}(t,x)) - J(t,x)]\right]$$

with

$$J(t,0) = \gamma\int_0^t \frac{\sigma(u)^2}{2} X(u)^2 du, \qquad \text{and} \qquad J(T,x) = Sx$$

Special Case II: Arrival Price Benchmark

This specific model corresponds to the case $X(\tau) = 0$ for all $\tau \in [0, T]$.

Theorem

The value function is the ***unique*** *solution of*

$$-\dot{J}(t,x) = \min\left[\inf_{0\le y\le x} -\dot{J}(t,x), \gamma\frac{\sigma(t)^2}{2}x^2 + \nu(t)\int_{[0,1]}[J(t,(x-y)^+) - J(t,x)]\nu_t(dy)\right]$$

with

$$J(t,0) = \gamma\int_0^t \frac{\sigma(u)^2}{2}X(u)^2 du, \qquad \text{and} \qquad J(T,x) = Sx$$

Special Case III: Stationary Approximation

When (t, x) is far enough from the corners $(0, 1)$ and $(T, 0)$, J looks like a function of $x - X(t)$ (**deviation from the benchmark**).

Stationarity assumption

- $\nu_1(dt) = \lambda dt$ for some constant $\lambda > 0$
- $\nu_t(dy) = \nu(dy)$ for all $t \in [0, T]$.
- $\sigma(t) = \sigma$ for all $t \in [0, T]$

Look for an approximation of the form

$$J(t, x) \approx \alpha + \beta x + w(x - X(t))$$

for some function w to be determined.

True in the **Large Fill case** (use the Lambert function)

The Discrete Case and Approximation Results

- The **integer** v denotes the quantity of shares (expressed as a number of lots) the broker has to sell by time T,
- Trades can only be in multiples of one lot.
- $t \hookrightarrow x_t$ looks like a staircase starting from $x_0 = 1$ and ending at $x_T = 0$.
- In units of v lots, the measures $\nu_t(dy)$ are supported by the grid $\{1/v, 2/v, \cdots, (v-1)/v, 1\}$
- The process $\underline{x} = (x_t)_{0 \le t \le T}$. and the controls $\underline{L} = (L_t)_{0 \le t \le T}$ and $\underline{m} = (m_t)_{0 \le t \le T}$ take values in the grid $\mathcal{I}_v := \{0, 1/v, \cdots, (v-1)/v, 1\}$.
- The sets of admissible controls are defined accordingly.

- Identify functions φ on the grid $\mathcal{I}_v$ with finite sequence $(\varphi_i)_{0 \le i \le v}$ where $\varphi_i = \varphi(i/v)$.
- Denote by $I\varphi$ the piecewise linear continuous function $[0, 1] \ni x \hookrightarrow [I\varphi](x)$ which coincides with φ on the grid $\mathcal{I}_v$ and which is linear on each interval $[i/v, (i+1)/v]$.
- $(\varphi_i)_{0 \le i \le v}$ is said to be convex if $I\varphi$ is convex
- For any integers v and v', and functions φ and φ' on the grids $\mathcal{I}_v$ and $\mathcal{I}_{v'}$, we have:

$$\|I\varphi - I\varphi'\|_\infty = \sup_{x \in [0,1]} |[I\varphi](x) - [I\varphi'](x)| = \sup_{x \in \mathcal{I}_v \cup \mathcal{I}_{v'}} |[I\varphi](x) - [I\varphi'](x)|.$$

Characterization of the Solution

The operators A and B become

$$[A\varphi]_i(t) = S - \varphi_i(t) + \varphi_{i-1}(t), \qquad i = 1, \cdots, v,$$

and

$$[B\varphi]_i(t) = \gamma \frac{\sigma(t)^2}{2}[X(t) - i/v]^2 + \nu(t) \min_{0 \le \ell \le i} \sum_{j=1}^{v} [\varphi_{i-j\wedge\ell}(t) - \varphi_i(t)]\nu_t(j/v)$$

so the HJB QVI remains the same:

$$\min\left[[A\varphi]_i(t), \dot{\varphi}_i(t) + [B\varphi]_i(t)\right] = 0, \qquad i = 1, \cdots, v.$$

As before we have existence and uniqueness of a C^1 functions of $t \in [0, T]$ satisfying

$$\varphi_i(t) = Si/v + \int_t^T \min_{0 \le j \le i} [B\varphi]_i(u)du, \qquad i = 0, 1, \cdots, v.$$

Interpreting the solution φ as a function on $[0, T] \times \mathcal{I}_v$ defined by $\varphi(t, i/v) = \varphi_i(t)$, since $\varphi_i(T) = Si/v$ and:

$$\dot{\varphi}_i(t) = -\min_{0 \le j \le i} [B\varphi]_j(t)$$

we get

$$\dot{\varphi}_i(t) + [B\varphi]_i(t) \ge 0, \qquad i = 0, 1, \cdots, v$$

and

$$\dot{\varphi}_i(t) = \max_{0 \le j \le i} \partial_t \varphi_j(t)$$

so that $i \hookrightarrow \dot{\varphi}_i(t)$ is non-decreasing and

$$-\dot{\varphi}_i(t) = \min\left[\min_{0 \le j \le i} -\dot{\varphi}_j(t), [B\varphi]_i(t)\right].$$

Characterization of the Value Function in the Discrete Case

Theorem

The value function J of the problem can be identified to the sequence $(J_i)_{0\leq 1\leq v}$ of C^1 functions of $t \in [0,T]$ satisfying:

$$\begin{cases} J_0(t) = \int_t^T \frac{\gamma\sigma(u)^2}{2} X(u)^2, \qquad J_i(T) = Si/v, \quad i = 0, 1, \cdots, v \\ \partial_t J_i(t) = \min \Big[\partial_t J_{i-1}(t), \\ \qquad \nu(t) \sum_{j=1}^{v} [\varphi_{(i-j)\vee\tilde{\ell}_i(t)}(t) - \varphi_i(t)]\nu_t(j) + \frac{\gamma\sigma(t)^2}{2}[X(t) - i/v]^2 \Big] \end{cases}$$

where

$$\tilde{\ell}_i(t) = \min\{\ell;\ \varphi_\ell(t) = \min_{0\leq j\leq i} \varphi_j(t)\}$$

Optimal Solution in the Discrete Case

- $\tau_i^m = \inf\{t \in [0, T];\ J_i(t) - J_{i-1}(t) < S/v\}$
- $\tau_i^l = \inf\{t \in [0, T];\ J_i(t) - J_{i-1}(t) < 0\}$
- $\tau_i^m \leq Tx(i - \frac{1}{2}) < \tau_i^l$