Optimal Execution:
II. Trade Optimal Execution

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Optimal Execution Set-Up

Goal: sell $x_0 > 0$ shares by time $T > 0$

- $\underline{X} = (X_t)_{0 \le t \le T}$ execution strategy
- X_t position (nb of shares held) at time t. $X_0 = x_0$, $X_T = 0$
- Assume X_t absolutely continuous (differentiable)
- $\tilde{P}_t$ **mid-price** (unaffected price), P_t **transaction price**, I_t **price impact**

$$P_t = \tilde{P}_t + I_t$$

  e.g. Linear Impact A-C model:

$$I_t = \gamma[X_t - X_0] + \lambda \dot{X}_t$$

- **Objective:** Maximize *form of revenue* at time T

Revenue $\mathcal{R}(\underline{X})$ from the execution strategy $\underline{X}$

$$\mathcal{R}(\underline{X}) = \int_0^T (-\dot{X}_t) P_t dt$$

Specific Challenges

- **First generation**: Price impact models (e.g. Almgren - Chriss)
  - Risk Neutral framework (maximize $\mathbb{E}\mathcal{R}(\underline{X})$) versus utility criteria
  - More complex portfolios (including options)
  - Robustness and performance constraints (e.g. slippage or tracking market VWAP)
- **Second generation**: Simplified LOB models
  - Simple liquidation problem
  - performance constraints (e.g. slippage or tracking market VWAP) and using **both** market and limit orders

Optimal Execution Problem in A-C Model

$$\begin{aligned}\mathcal{R}(\underline{X}) &= \int_0^T (-\dot{X}_t) P_t dt \\ &= -\int_0^T \dot{X}_t \tilde{P}_t dt - \int_0^T \dot{X}_t I_t dt \\ &= x_0 \tilde{P}_0 + \int_0^T X_t d\tilde{P}_t - \mathcal{C}(\underline{X})\end{aligned}$$

with $\mathcal{C}(\underline{X}) = \int_0^T \dot{X}_t I_t dt$. Interpretation

- $x_0 \tilde{P}_0$ (initial) **face value** of the portfolio to liquidate
- $\int_0^T X_t d\tilde{P}_t$ **volatility risk** for selling according to $\underline{X}$ instead of immediately!
- $\mathcal{C}(\underline{X})$ **execution costs** due to market impact

Special Case: the Linear A-C Model

$$\mathcal{R}(\underline{X}) = x_0 \tilde{P}_0 + \int_0^T X_t d\tilde{P}_t - \lambda \int_0^T \dot{X}_t^2 dt - \frac{\gamma}{2} x_0^2$$

Easy Case: Maximizing $\mathbb{E}[\mathcal{R}(X)]$

$$\mathbb{E}[\mathcal{R}(\underline{X})] = x_0 P_0 - \frac{\gamma}{2} x_0^2 - \lambda \mathbb{E} \int_0^T \dot{X}_t^2 dt$$

Jensen's inequality & constraints $X_0 = x_0$ and $X_T = 0$ imply

$$\dot{X}_t^* = -\frac{x_0}{T}$$

trade at a constant rate **indpdt of volatility** ! **Bertsimas - Lo (1998)**

More Realistic Problem

Almgren - Chriss propose to maximize

$$\mathbb{E}[\mathcal{R}(\underline{X})] - \alpha \mathrm{var}[\mathcal{R}(X)]$$

(α risk aversion parameter – late trades carry volatility risk)

For **DETERMINISTIC** trading strategies $\underline{X}$

$$\mathbb{E}[\mathcal{R}(\underline{X})] - \alpha \mathrm{var}[\mathcal{R}(X)] = x_0 P_0 - \frac{\gamma}{2} x_0^2 - \int_0^T \left(\frac{\alpha\sigma^2}{2} X_t^2 + \lambda \dot{X}_t^2 \right) dt$$

maximized by (standard variational calculus with constraints)

$$\dot{X}_t^* = x_0 \frac{\sinh \kappa (T - t)}{\sinh \kappa T} \qquad \text{for} \qquad \kappa = \sqrt{\frac{\alpha \sigma^2}{2\lambda}}$$

For **RANDOM** (adapted) trading strategies $\underline{X}$, **more difficult** as
Mean-Variance not amenable to dynamic programming

Maximizing Expected Utility

Choose $U : \mathbb{R} \to \mathbb{R}$ increasing concave and

$$\text{maximize} \qquad \mathbb{E}[U(\mathcal{R}(\underline{X}_T)]$$

Stochastic control formulation over a state process $(X_t, R_t)_{0 \le t \le T}$.

$$v(t,x,r) = \sup_{\underline{\xi} \in \Xi(t,x)} \mathbb{E}[u(R_T)|X_t = x, R_T = r]$$

value function, where $\Xi(t,x)$ is the set of **admissible controls**

$$\left\{ \underline{\xi} = (\xi_s)_{t \le s \le T}; \text{progressively measurable}, \int_t^T \xi_s^2 ds < \infty, \int_t^T \xi_s ds = x \right\}$$

$$X_s = X_s^{\underline{\xi}} = x - \int_t^s \xi_u du, \qquad \dot{X}_s = -\xi_s,\ X_t = x$$

and (choosing $\tilde{P}_t = \sigma W_t$)

$$R_s = R_s^{\underline{\xi}} = R + \sigma \int_t^s X_u dW_u - \lambda \int_t^s \xi_u^2 du, \quad dR_s = \sigma X_s dW_s - \lambda \xi_s^2 ds,\ R_t = r$$

Finite Fuel Problem

Non Standard Stochastic Control problem because of the constraints

$$\int_0^T \xi_s ds = x_0.$$

Still, one expects

- For any admissible $\underline{\xi}$, $[v(t, X_t^{\xi}, R_t^{\xi})]_{0 \le t \le T}$ is a super-martingale
- For some admissible $\underline{\xi}^*$, $[v(t, X_t^{\xi^*}, R_t^{\xi^*})]_{0 \le t \le T}$ is a **true** martingale

If v is smooth, and we set $V_t = v(t, X_t^{\xi}, R_t^{\xi})$, Itô's formula gives

$$\begin{aligned} dV_t &= \Big(\partial_t v(t, X_t, R_t) + \frac{\sigma^2}{2}\partial^2_{rr} v(t, X_t, R_t) \\ &\qquad -\lambda \xi_t^2 \partial_r v(t, X_t, R_t) - \xi_t \partial_x v(t, X_t, R_t)\Big) dt \\ &\qquad + \sigma \partial_x v(t, X_t, R_t) dW_t \end{aligned}$$

Hamilton-Jabobi-Bellman Equation

One expects that v solves the **HJB** equation (nonlinear PDE)

$$\partial_t v + \frac{\sigma^2}{2}\partial^2_{xx} v - \inf_{\xi \in \mathbb{R}} [\xi^2 \lambda \partial_r v + \xi \partial_x v] = 0$$

in some sense, with the (non-standard) terminal condition

$$v(T, x, r) = \begin{cases} U(r) & \text{if } x = X_0 \\ -\infty & \text{otherwise} \end{cases}$$

Solution for CARA Exponential Utility

For $u(x) = -e^{-\alpha x}$ and κ as before

$$v(t, x, r) = e^{-\alpha r + x_0^2 \alpha \lambda \kappa \coth \kappa (T-t)}$$

solves the HJB equation and the unique maximizer is given by the
DETERMINISTIC

$$\xi_t^* = x_0 \kappa \frac{\cosh \kappa (T-t)}{\sinh \kappa T}$$

Schied-Schöneborn-Tehranchi (2010)

- Optimal solution **same** as in Mean - Variance case
- **Schied-Schöneborn-Tehranchi**'s trick shows that optimal trading strategy is **generically** deterministic for **exponential utility**
- **Open problem** for general utility function
- Partial results in **infinite horizon** versions

Shortcomings

- Optimal strategies
  - are **DETERMINISTIC**
  - do not react to **price changes**
  - are **time inconsistent**
  - are **counter-intuitive** in some cases
- Computations require
  - solving **nonlinear PDEs**
  - with **singular** terminal conditions

Recent Developments

Gatheral - Schied (2011), Schied (2012)

- In the spirit of Almgren-Chriss mean-variance criterion, maximize

$$\mathbb{E}\left[\mathcal{R}(\underline{X}) - \tilde{\lambda}\int_0^T X_t P_t dt\right]$$

- The solution happens to be **ROBUST**
  - $\tilde{P}_t$ can be a semi-martingale, optimal solution does not change

Recent Developments

Almgren - Li (2012), Hedging a large option position

- $g(t, \tilde{P}_t)$ price at time t of the option (from Black-Scholes theory)
- Revenue

$$\mathcal{R}(\underline{X}) = g(T, \tilde{P}_T) + X_T \tilde{P}_T - \int_0^T \tilde{P}_t \dot{X}_t dt - \lambda \int_0^T \dot{X}_t^2 dt$$

- Using Itô's formula and the fact that g solves a PDE,

$$\mathcal{R}(\underline{X}) = R_0 + \int_0^T [X_t + \partial_x g(t, \tilde{P}_t)] dt - \lambda \int_0^T \dot{X}_t^2 dt \qquad R_0 = x_0 \tilde{P}_0 + g(0, \tilde{P}_0)$$

- **Introduce** $Y_t = X_t + \partial_x g(t, \tilde{P}_t)$ for **hedging correction**

$$\begin{cases} d\tilde{P}_t = \gamma \dot{X}_t dt + \sigma dW_t \\ dY_t = [1 + \gamma \partial^2_{xx} g(t, \tilde{P}_t)] dt + \sigma \partial^2_{xx} g(t, \tilde{P}_t) dW_t \end{cases}$$

- **Minimize**

$$\mathbb{E}\left[G(\tilde{P}_T, Y_T) + \int_0^T \left(\frac{\sigma^2}{2} Y_t^2 - \gamma \dot{X}_t Y_t + \lambda \dot{X}_t^2 \right) dt \right]$$

Explicit solution in some cases (e.g. $\partial^2_{xx} g(t, x) = c$, G quadratic)

Transient Price Impact

Flexible price impact model

- **Resilience function** $G : (0, \infty) \to (0, \infty)$ measurable bounded
- Admissible $\underline{X} = (X_t)_{0 \le t \le T}$ cadlag, adapted, **bounded variation**
- Transaction price

$$P_t = \tilde{P}_t + \int_0^t G(t-s)\, dX_s$$

- Expected cost of strategy $\underline{X}$ given by

$$-x_0 P_0 + \mathbb{E}[\mathcal{C}(\underline{X})]$$

where

$$\mathcal{C}(\underline{X}) = \int \int G(|t-s|) dX_s dX_t$$

Transient Price Impact: Some Results

- No **Price Manipulation** in the sense of **Huberman - Stanzl (2004)** if $G(|\cdot|)$ positive definite
- Optimal strategies (if any) are **deterministic**
- Existence of an optimal $\underline{X}^*$ $\Leftrightarrow$ solvability of a Fredholm equation
- Exponential Resilience $G(t) = e^{-\rho t}$

$$dX_t^* = -\frac{x_0}{\rho T + 2}\Big(\delta_0(dt) + \rho dt + \delta_T(dt)\Big)$$

- $\underline{X}^*$ purely discrete measure on $[0, T]$ when $G(t) = (1 - \rho t)^+$ with $\rho > 0$
  - $dX_t^* = -\frac{x_0}{2}[\delta_0(dt) + \delta_T(dt)]$ if $\rho < 1/T$
  - $dX_t^* = -\frac{x_0}{n+1}\sum_{i=0}^n \delta_{iT/n}(dt)$ if $\rho < n/T$ for some integer $n \geq 1$

Obizhaeva - Wang (2005), Gatheral - Schied (2011)

Optimal Execution in a LOB Model

- Unaffected price $\tilde{P}_t$ (e.g. $\tilde{P}_t = P_0 + \sigma W_t$)
- Trader places only **market sell orders**
  - Placing buy orders is not optimal
- Bid side of LOB given by a function $f : \mathbb{R} \to (0, \infty)$ s.t. $\int_0^\infty f(x)dx = \infty$. At any time t

$$\int_a^b f(x)dx = \text{bids available in the price range } [\tilde{P}_t + a, \tilde{P}_t + b]$$

- The **shape function** f does not depend upon t or $\tilde{P}_t$

Obizhaeva - Wang (2006), Alfonsi - Schied - Schulz (2011),Predoiu - Shaikhet - Shreve (2011)

Optimal Execution in a LOB Model (cont.)

- **Price Impact** process $\underline{D} = (D_t)_{0 \le t \le T}$ adapted, cadlag

  At time t a market order of size A moves the price from $\tilde{P}_t + D_{t-}$ to $\tilde{P}_t + D_t$ where

$$\int_{D_{t-}}^{D_t} f(x)dx = A$$

- **Volume Impact** $Q_t = F(D_t)$ where $F(x) = \int_0^x f(x')dx'$.
- LOB **Resilience**: Q_t and D_t decrease between trades, e.g.

$$dQ_t = -\rho Q_t dt, \qquad \text{for some } \rho > 0$$

- At time t, a sell of size A will bring

$$\begin{aligned} \int_{D_{t-}}^{D_t} (\tilde{P}_t + x) f(x)dx &= A\tilde{P}_t + \int_{D_{t-}}^{D_t} x dF(x) \\ &= A\tilde{P}_t + \int_{Q_{t-}}^{Q_t} \psi(x)dx = A\tilde{P}_t + \Psi(Q_t) - \Psi(Q_{t-}) \end{aligned}$$

  if $\psi = F^{-1}$ and $\Psi(x) = \int_0^x \psi(x')dx'$.

Stochastic Control Formulation

Holding trajectories / Trading strategies

$$\Xi(t,x) = \left\{ (\Xi_s)_{t\le s\le T} : \ \text{càdlàg, adapted, bounded variation, } \Xi_t = x \right\}$$

$$\Xi_{ac}(t,x) = \left\{ (\Xi_s)_{t\le s\le T} : \ \Xi_s = x + \int_t^s \xi_r dr \text{ for } (\xi_s)_{t\le s\le T} \text{ bounded adapted } \right\}$$

$$\begin{cases} dX_t &= -d\Xi_t \\ dQ_t &= -d\Xi_t - \rho Q_t dt \\ dR_t &= -\rho Q_t \psi(Q_t) dt - \sigma \Xi_t dW_t \end{cases}$$

Value Function Approach

State space process $Z_t = (X_t, Q_t, R_t)$, value function

$$v(t, x, q, r) = v(t, z) = sup_{\xi \in \Xi(t,x)} \mathbb{E}[U(R_T - \Psi(Q_T)]$$

First properties

- $U(r - \Psi(q + r)) \leq v(t, x, q, r) \leq U(r - \Psi(q))$
- $v(t, x, q, r) = U(r - \Psi(q + r))$ for $x = 0$ and $t = T$
- Functional approximation arguments imply

$$\begin{aligned} v(t, x, q, r) &= \sup_{\xi \in \Xi(t,x)} \mathbb{E}[U(R_T - \Psi(Q_T)] \\ &= \sup_{\xi \in \Xi_{ac}(t,x)} \mathbb{E}[U(R_T - \Psi(Q_T)] \\ &= \sup_{\xi \in \Xi_{d}(t,x)} \mathbb{E}[U(R_T - \Psi(Q_T)] \end{aligned}$$

QVI Formulation

As before

- Assume v smooth and apply Itô's formula to $v(t, X_t, Q_t, R_t)$
- $v(t, X_t, Q_t, R_t)$ is a super-martingale for a typical $\underline{\xi}$ implies

$$\partial_t v + \frac{\sigma^2}{2} x^2 \partial^2_{rr} v - \rho q \psi(q) \partial_r v - \rho q \partial_q v \geq 0$$

- $\partial_x v - \partial_q v \geq 0$

QVI (Quasi Variational Inequality) instead of **HJB** nonlinear PDE

$$\min[\partial_t v + \frac{\sigma^2}{2} x^2 \partial^2_{rr} v - \rho q \psi(q) \partial_r v - \rho q \partial_q v, \partial_x v - \partial_q v] = 0$$

with terminal condition $v(T, x, q, r) = U(r - \Psi(x + q))$

Existence and Uniqueness of a *viscosity solution*
R.C. - H. Luo (2012)

Special Cases

Assuming a flat LOB $f(x) = c$ and $U(c) = x$

$$v(t,x,q,r) = r - \frac{q^2(1-e^{-2\rho s})}{2c} - \frac{(x+qe^{-}\rho s)^2}{c(2+\rho(T-t-s)}$$

with $s = (T-t) \wedge \inf\{u \in [0,T];\ (1+\rho(T-t-u))qe^{-\rho u} \leq x\}$

Still with $f(x) = c$ but for a CARA utility $U(x) = -e^{-\alpha x}$

$$v(t,x,q,r) = -\exp\left[-\alpha r - \frac{\alpha}{2c}(\alpha c\sigma^2 xx^2 + q^2(1-e^{-2\rho s}) + \varphi(t+s)(x+qe^{-\rho s})^2\right]$$

where φ is the solution of the Ricatti's equation

$$\dot{\varphi}(t) = \frac{\rho^2}{2\rho+\alpha c\sigma^2}\varphi(t)^2 + \frac{2\rho\alpha c\sigma^2}{2\rho+\alpha c\sigma^2}\varphi(t) - \frac{2\rho\alpha c\sigma^2}{2\rho+\alpha c\sigma^2}, \qquad \varphi(T) = 1$$

and
$s = (T-t) \wedge \inf\{u \in [0,T];\ (\alpha c\sigma^2 + \rho\varphi(t+u))x \geq \rho(2-\varphi(t+u))qe^{-\rho u}\}$