Optimal Execution:
III. Game Theory & Predatory Trading

René Carmona

Bendheim Center for Finance
Department of Operations Research & Financial Engineering
Princeton University

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Premises for Predatory Trading

- Large Trader facing a Forced Liquidation
- Especially if the need to liquidate is known by other traders
  - hedge funds with (nearing) margin call
  - traders who use portfolio insurance, stop loss orders, . . .
  - some institutions / funds cannot hold on to downgraded instruments
  - Index-replication funds (at re-balancing dates) e.g. Russell 3000

Forced liquidation can be **very costly** because of **price impact**

Business Week

> *...if lenders know that a hedge fund needs to sell something quickly, they will sell the same asset, driving the price down even faster. Goldman Sachs and other counter-parties to LTCM did exactly that in 1998.*

> *When you smell blood in the water, you become a shark . . . . when you know that one of your number is in trouble . . . you try to figure out what he owns and you start shorting those stocks . . .*

Cramer (2002)

Typical Predatory Trading Scenario

- Distressed trader needs to unload a large position
- Size will have impact on price
- Predator initially trades in the same direction as the prey
- Effect is to withdraw liquidity
- Market impact of the liquidation becomes greater
- Price fall is exaggerated (**over-shooting**)
- Predator reverses direction, profiting from the over-shoot
- Predator closes position for a profit.

Optimal Portfolio Liquidation: Multi-Player Case

- New issues when other market participants know that our client is selling:
- The market impact of our client creates a drift in the market price
- This drift can be exploited by the other market participants
- Since we know about this danger, we will adjust our strategy
  - Brunnermeier and Pedersen (2005),
  - Carlin, Lobo, and Viswanathan (2005)
  - SChied, Schöneborn (2008)

Game Model

- One risk free asset and one risky asset
- Trading in continuous time, interest rate $r = 0$
- $n+1$ strategic players and a number of noise traders
- $X_0(t), X_1(t), \cdots, X_n(t)$ risky asset positions of the strategic players
- Trades at time t are executed at the price (**Chriss-Almgren price impact model**)

$$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} [X_i(t) - X_i(0)] + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

where $\tilde{P}(t)$ is a **mean zero** martingale (say a Wiener process).

Goal of the Mathematical Analysis

- Understand predation
- Illustrate benefits of
  - Stealth trading
  - Sunshine trading

Modeling extreme markets

- **Elastic** (truly illiquid) markets:
  - temporary impact $\lambda >> $ permanent impact γ
- **Plastic** (nervous) markets:
  - permanent impact $\gamma >> $ temporary impact λ

Assumptions of the One Period Game

- Each strategic player $i \in \{0, 1, \cdots, n\}$ knows
  - all other strategic players initial asset positions $X^j(0)$
  - Their target $X_j(T)$ at some fixed time point $T > 0$ in the future
- Objective (all players are **risk neutral**)
  - Players maximize their expected return by choosing an optimal trading strategy $X_i(t)$ satisfying their constraints $X_i(0)$ and $X_i(T)$

One **distressed trader** / **prey** (e.g **seller**), player 0

$$X_0(0) = x_0 > 0, \qquad X_0(T) = 0$$

n **predators** players $1, 2, \cdots, n$

$$X_i(0) = X_i(T) = 0, \qquad i = 1, \cdots, n$$

Optimization Problem

A strategy $X_i = (X_i(t))_{0 \leq t \leq T}$ is **admissible** (for player i) if it is an a

- adapted process
- with continuously differentiable sample paths

Given a set $\underline{X} = (X_0, X_1, \cdots, X_n)$ of admissible strategies

- Each player $i \in \{0, 1, \cdots, n\}$ tries to maximize his expected return

  $$J^i(\underline{X}) = \mathbb{E}[\int_0^T (-\dot{X}_i(t))P(t)dt]$$

  under the constraint

  $$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} [X_i(t) - X_i(0)] + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

- Search for **Nash Equilibrium**

Deterministic Strategies

If we restrict the admissible strategies $\underline{X} = (X_0, X_1, \cdots, X_n)$ to be **DETERMINISTIC**

$$J^i(\underline{X}) = \mathbb{E}[\int_0^T (-\dot{X}_i(t))P(t)dt] = \mathbb{E}[\int_0^T (-\dot{X}_i(t))\overline{P}(t)dt]$$

where

$$\overline{P}(t) = P(0) + \gamma \sum_{i=0}^{n} [X_i(t) - X_i(0)] + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

THE SOURCE OF RANDOMNESS IS GONE !

Carlin, Lobo, and Viswanathan (2005) Schied, Schoenborn (2008)

Solution in the Deterministic Case

Unique Optimal Strategies

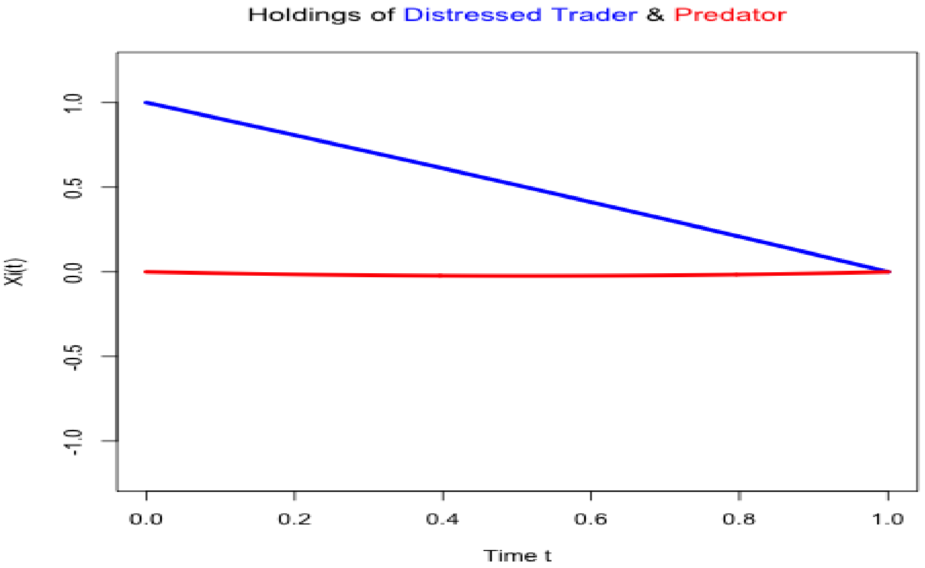
$$X_i(t) = ae^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} + b_i e^{\frac{\gamma}{\lambda}t}$$

where

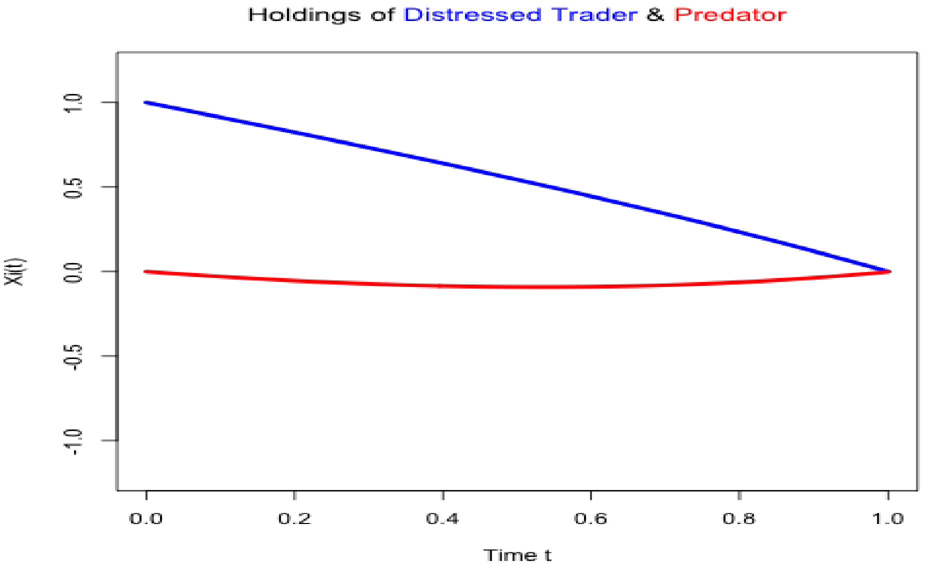
$$\begin{aligned}
a &= \frac{n}{n+2}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T}\right)^{-1}\frac{1}{n+1}\sum_{i=0}^{n}[X_i(T) - X_i(0)] \\
b_i &= \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1}\left(X_i(T) - X_i(0) - \frac{1}{n+1}\sum_{i=0}^{n}[X_i(T) - X_i(0)]\right)
\end{aligned}$$

Carlin, Lobo, and Viswanathan (2005)

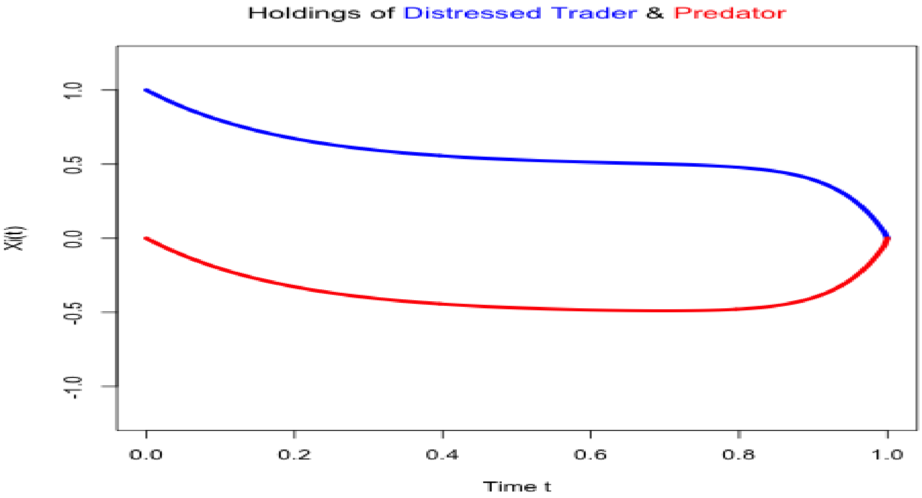
$n = 1$ predator, $\gamma/\lambda = 0.3$



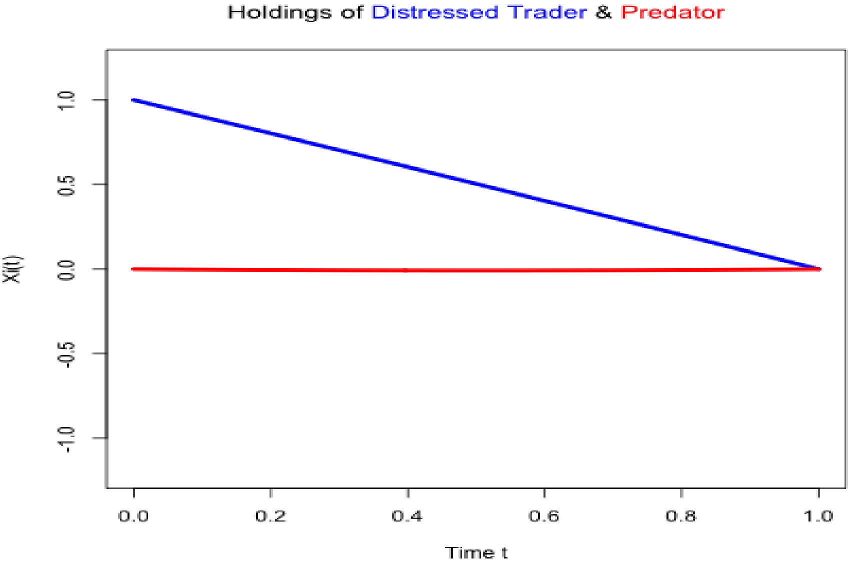
$n = 1$ predator, $\gamma = \lambda$



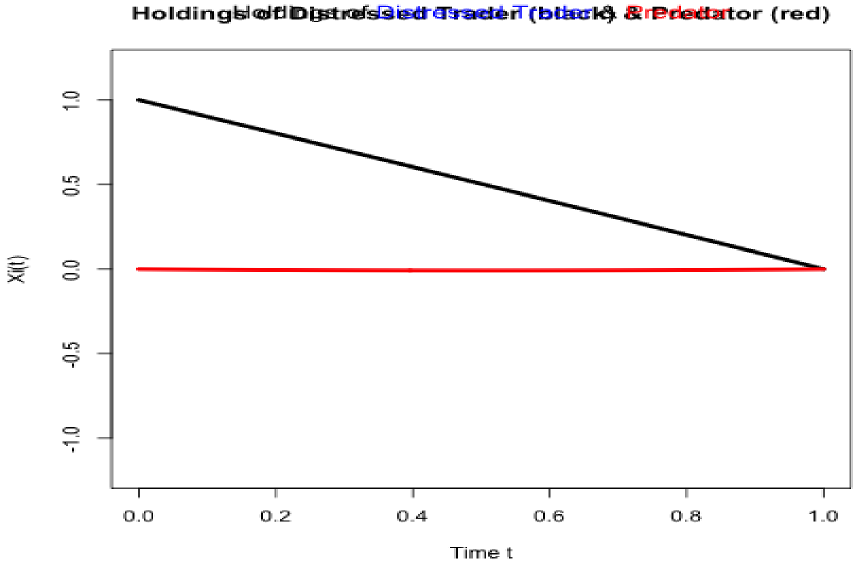
$n = 1$ predator, $\gamma = 15.5\lambda$



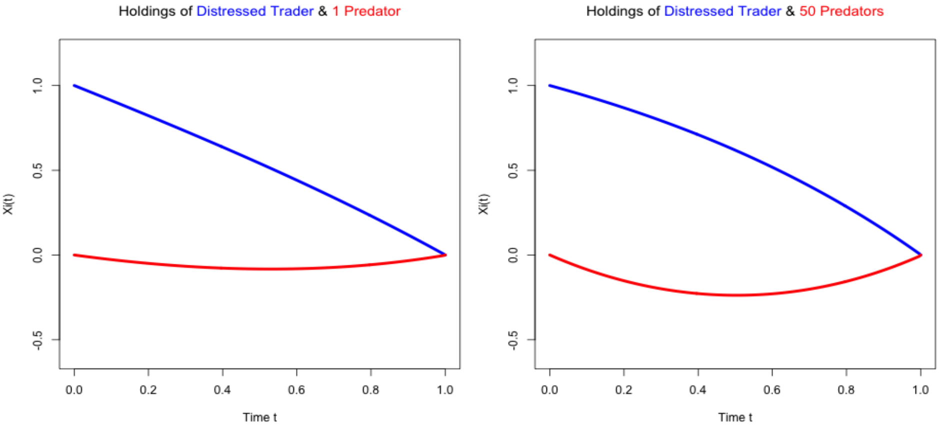
Holdings of the Distressed Trader & Predator



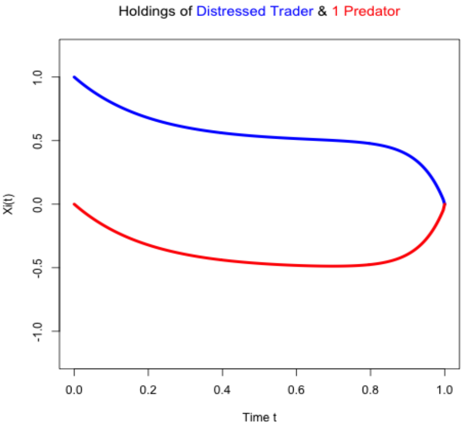
Fancy Plots of the Holdings of the Distressed Trader & Predator

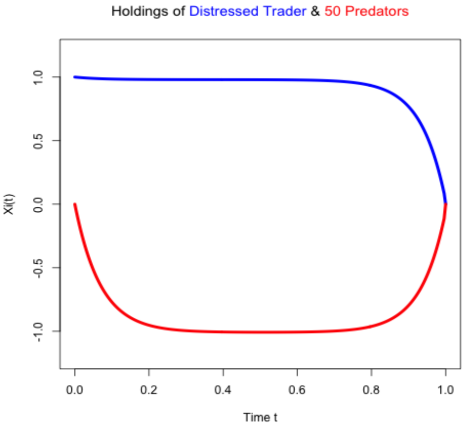


Impact of the Number of Predators: $\gamma = \lambda$

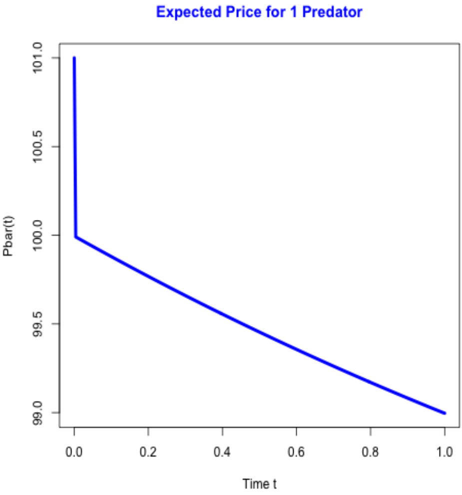


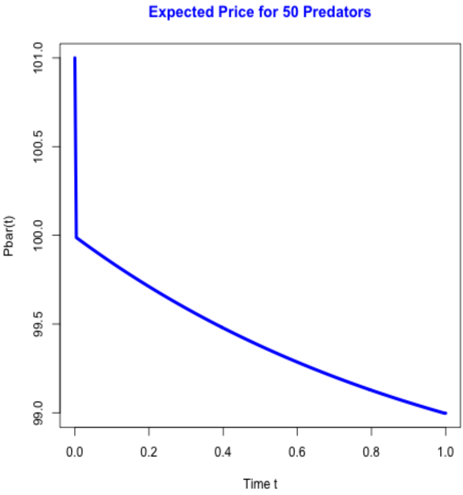
Impact of the Number of Predators: $\gamma = 15.5\lambda$



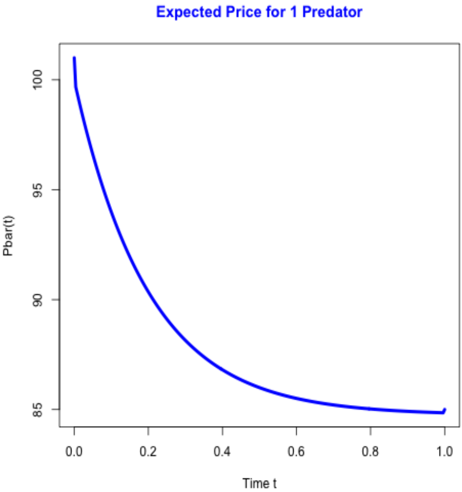


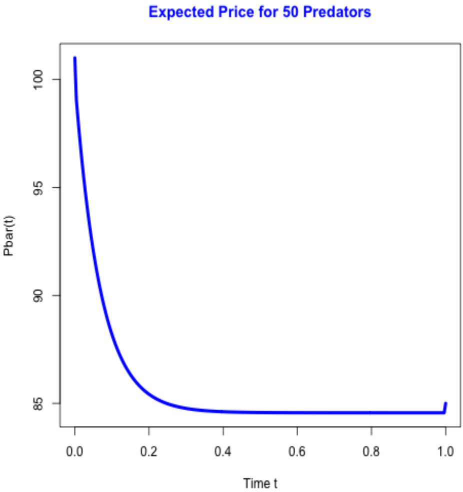
Expected Price: $\gamma = \lambda$



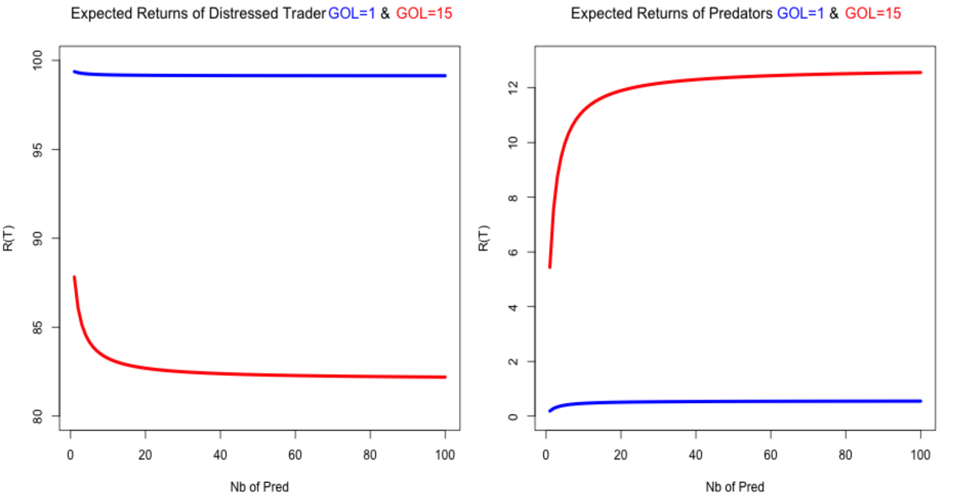


Expected Price: $\gamma = 15\lambda$





Impact of Nb of Predators on Expected Returns



Two Period Model

- Prey has to liquidate $X_0 > 0$ by time T_1, i.e. $X_0(T_1) = 0$
- Predators can stay in the game longer $X_i(0) = X_i(T_2) = 0$ for some $T_2 > T_1$ for $i = 1, \cdots, n$
- Prey does not trade in second period $[T_1, T_2]$, i.e. $X_0(t) = 0$ for $T_1 \leq t \leq T_2$.

Markovian Structure $\Longrightarrow$

Solution determined by predators' positions at time T_1

Nash Equilibrium for Deterministic Strategies

UNIQUE Nash Equilibrium

- **ALL** Predators have the same position at time T_1

$$X_i(T_1) = \frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0} X_0, \qquad i = 1, \cdots, n$$

- Coefficients depend upon n but converge as $n \to \infty$
- Asymptotic formulas for expected returns
- Asymptotic comparison of **Stealth** versus Sunshine trading for some regimes of γ/λ

Schöneborn - Schied (2008)