Energy Markets II: Spread Options, Weather Derivatives & Asset Valuation

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Banff, May 2007

European Call on the difference between two indexes

Calendar Spread Options

- Single Commodity at two different times

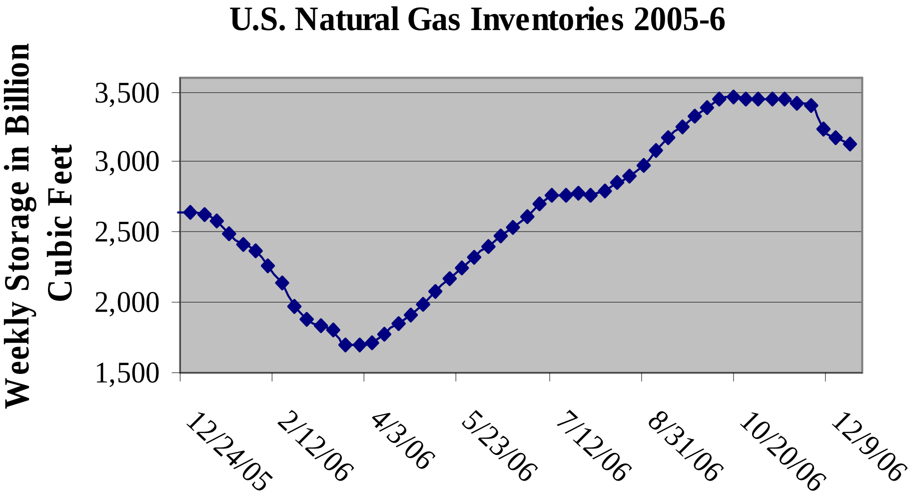
$$\mathbb{E}\{(I(T_2) - I(T_1) - K)^+\}$$

- Mathematically easier (only one underlier)

European Call on the difference between two indexes

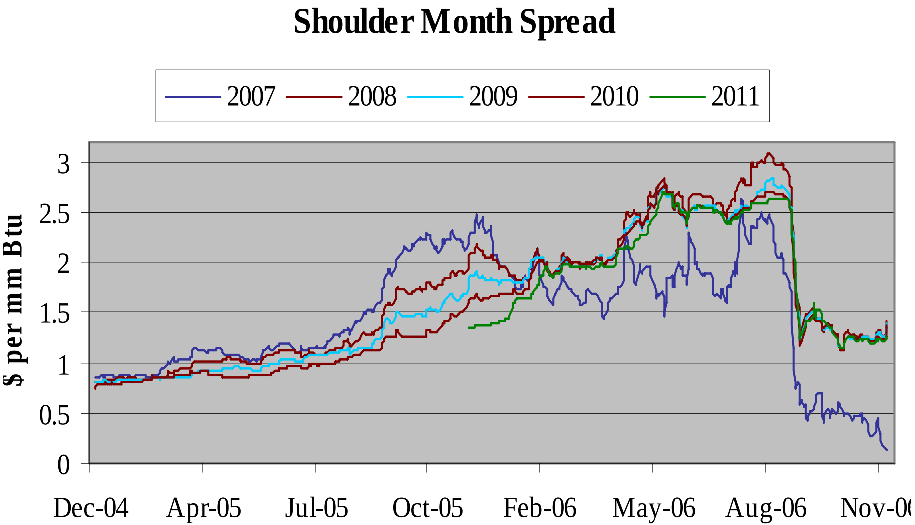
- **Calendar Spread**
- **Amaranth** largest (and **fatal**) positions
  - Shoulder Natural Gas Spread (play on inventories)
  - **Long** March Gas
  - **Short** April Gas
    - Depletion stops in March, injection starts in April
    - Can be fatal: emph**widow maker spread**

Seasonality of Gas Inventory



(from Raj Hatharamani ORFE Senior Thesis)

What Killed Amaranth



(from Raj Hatharamani ORFE Senior Thesis)

More Spread Options

- **Cross Commodity**
  - Crush Spread: between Soybean and soybean products (meal & oil)
  - Crack Spread:
    - gasoline crack spread between Crude and Unleaded
    - heating oil crack spread between Crude and HO
  - **Spark spread**

$$S_t = F_E(t) - H_{eff} F_G(t)$$

H_{eff} **Heat Rate**

Synthetic Generation

Present value of profits for future power generation (case of one fuel)

$$\mathbb{E}\{ \int_0^T D(0,t)(\tilde{F}_P(t,\tau) - H * \tilde{F}_G(t,\tau) - K)^+ \, dt\}$$

where

- $\tau > 0$ fixed (small)
- $D(0,t)$ **discount factor** to compute present values
- $\tilde{F}_P(t,\tau)$ (resp. $\tilde{F}_G(t,\tau)$) price at time t of a power (resp. gas) contract with delivery $t+\tau$
- H **Heat Rate**
- K **Operation and Maintenance** cost (sometimes denoted $O\&M$)

Basket of Spread Options

Deterministic discounting (with constant interest rate)

$$D(t,T) = e^{-r(T-t)}$$

Interchange **expectation** and **integral**

$$\int_0^T e^{-rt}\mathbb{E}\{(\tilde{F}_P(t,\tau) - H * \tilde{F}_G(t,\tau) - K)^+\}\, dt$$

Continuous **stream of spread options**

In Practice

- **Discretize time**, say daily

$$\sum_{t=0}^{T} e^{-rt}\mathbb{E}\{(\tilde{F}_P(t,\tau) - H * \tilde{F}_G(t,\tau) - K)^+$$

- **Bin** Daily Production in **Buckets** B_k's (e.g. 5×16, 2×16, 7×8, settlement locations, ...).

$$\sum_{t=0}^{T} e^{-r(T-t)} \sum_k \mathbb{E}\{(\tilde{F}_P^{(k)}(t,\tau) - H^{(k)} * \tilde{F}_G^{(k)}(t,\tau) - K^{(k)})^+\}$$

Basket of Spark Spread Options

$$p = e^{-rT}\mathbb{E}\{(I_2(T) - I_1(T) - K)^+\}$$

- Underlying indexes are spot prices
  - Geometric Brownian Motions ($K = 0$ Margrabe)
  - Geometric Ornstein-Uhlembeck (OK for Gas)
  - Geometric Ornstein-Uhlembeck with jumps (OK for Power)
- Underlying indexes are forward/futures prices
  - HJM-type models with deterministic coefficients

Problem

finding closed form formula and/or fast/sharp approximation for

$$\mathbb{E}\{(\alpha e^{\gamma X_1} - \beta e^{\delta X_2} - \kappa)^+\}$$

for a Gaussian vector (X_1, X_2) of $N(0,1)$ random variables with correlation ρ.

Sensitivities?

Easy Case : Exchange Option & Margrabe Formula

$$p = e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T))^+\}$$

- $S_1(T)$ and $S_2(T)$ **log-normal**
- p given by a formula *à la Black-Scholes*

$$p = x_2\Phi(d_1) - x_1\Phi(d_0)$$

with

$$d_1 = \frac{\ln(x_2/x_1)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \qquad d_0 = \frac{\ln(x_2/x_1)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}$$

and:

$$x_1 = S_1(0), \quad x_2 = S_2(0), \quad \sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

- Deltas are also given by "closed form formulae".

Proof of Margrabe Formula

$$p = e^{-rT}\mathbb{E}_{\mathbb{Q}}\{(S_2(T) - S_1(T))^+\} = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left\{\left(\frac{S_2(T)}{S_1(T)} - 1\right)^+ S_1(T)\right\}$$

- $\mathbb{Q}$ risk-neutral probability measure
- Define (Girsanov) $\mathbb{P}$ by:

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = S_1(T) = \exp\left(-\frac{1}{2}\sigma_1^2 T + \sigma_1 \hat{W}_1(T)\right)$$

- Under $\mathbb{P}$,
  - $\hat{W}_1(t) - \sigma_1 t$ and $\hat{W}_2(t)$
  - S_2/S_1 is geometric Brownian motion under $\mathbb{P}$ with volatility

$$\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

$$p = S_1(0)\mathbb{E}_{\mathbb{P}}\left\{\left(\frac{S_2(T)}{S_1(T)} - 1\right)^+\right\}$$

Black-Scholes formula with $K = 1$, σ as above.

(Classical) Real Option Power Plant Valuation

Real Option Approach

- Lifetime of the plant $[T_1, T_2]$
- C **capacity** of the plant (in MWh)
- H **heat rate** of the plant (in MMBtu/MWh)
- P_t price of **power** on day t
- G_t price of **fuel** (gas) on day t
- K fixed **Operating Costs**
- **Value of the Plant (ORACLE)**

$$C \sum_{t=T_1}^{T_2} e^{-rt} \mathbb{E}\{(P_t - HG_t - K)^+\}$$

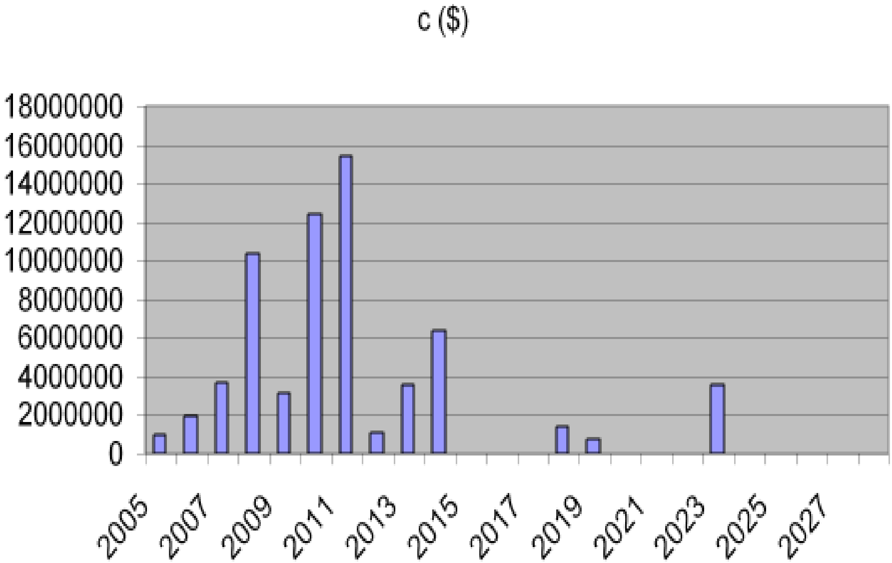
String of Spark Spread Options

(Flash Back)

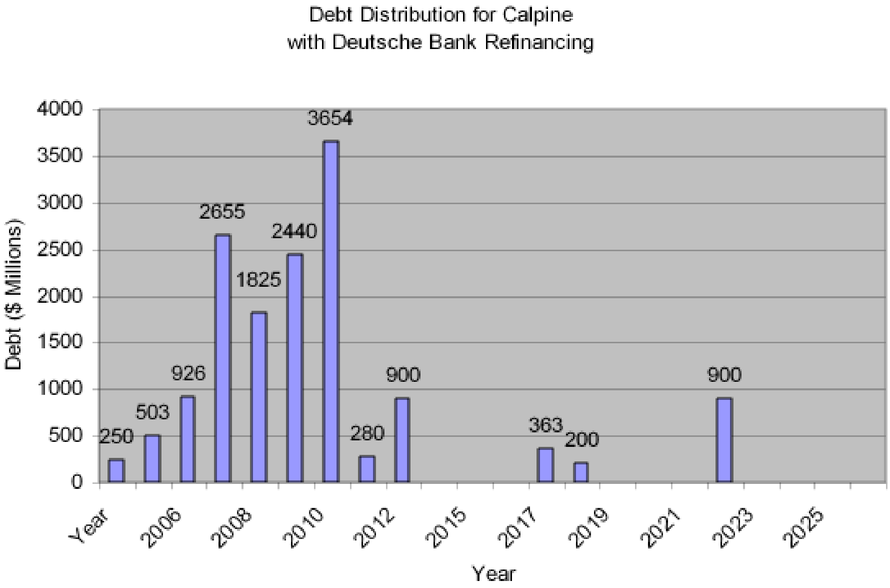
The Calpine - Morgan Stanley Deal

- Calpine needs to refinance USD 8 MM by November 2004
- **Jan. 2004**: Deutsche Bank: no traction on the offering
- **Feb. 2004**: *The Street* thinks Calpine is ”heading South”
- **March 2004**: Morgan Stanley offers a (complex) structured deal
  - A strip of spark spread options on 14 Calpine plants
  - A similar bond offering
- ***How were the options priced?***
  - By Morgan Stanley ?
  - By Calpine ?

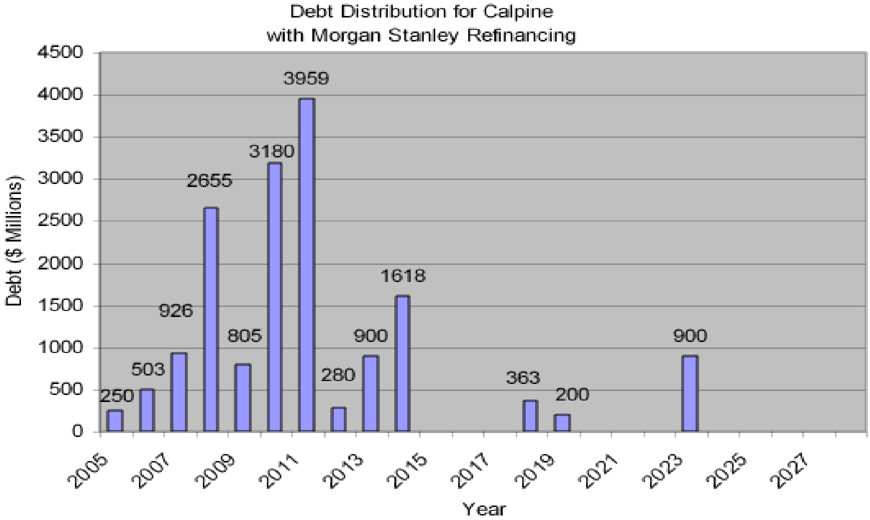
Calpine Debt



Calpine Debt with Deutsche Bank Financing



Calpine Debt with Morgan Stanley Financing



A Possible Model

Assume that Calpine owns **only** one plant

MS guarantees its spark spread will be at least κ for M years

Approach à la **Leland**'s Theory of the **Value of the Firm**

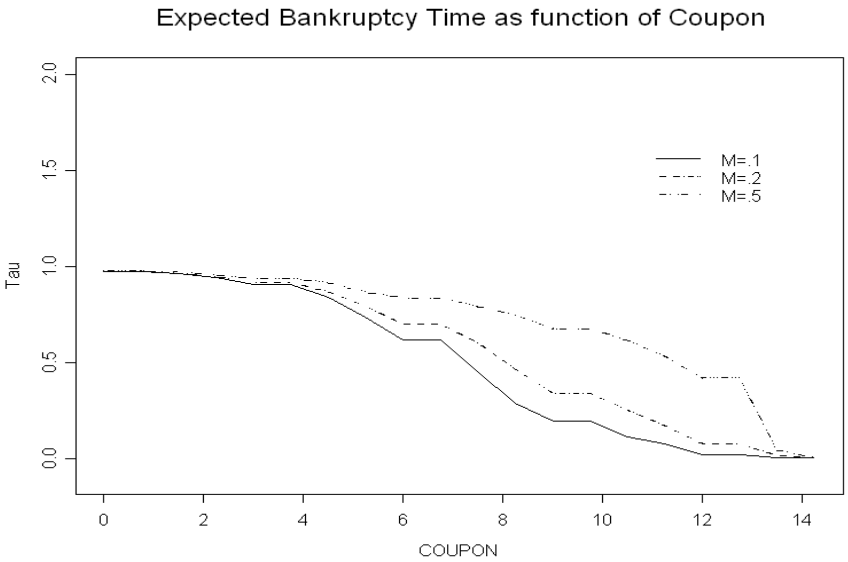
$$V = v - p_0 + \sup_{\tau \le T} \mathbb{E}\left\{ \int_0^\tau e^{-rt} \overline{\delta}_t \, dt \right\}$$

where

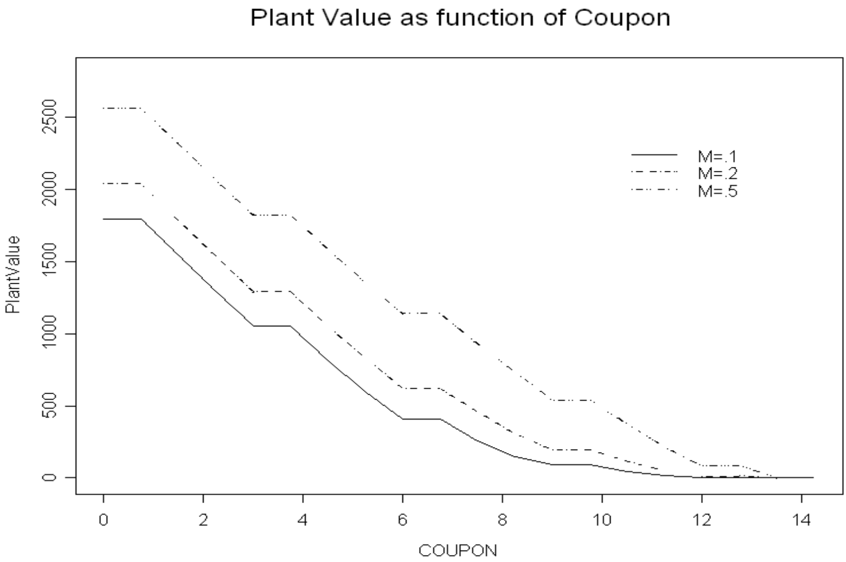
$$\overline{\delta}_t = \begin{cases} (P_t - H * G_t - K) \vee \kappa - c_t & \text{if } 0 \le t \le M \\ (P_t - H * G_t - K)^+ - c_t & \text{if } M \le t \le T \end{cases}$$

and

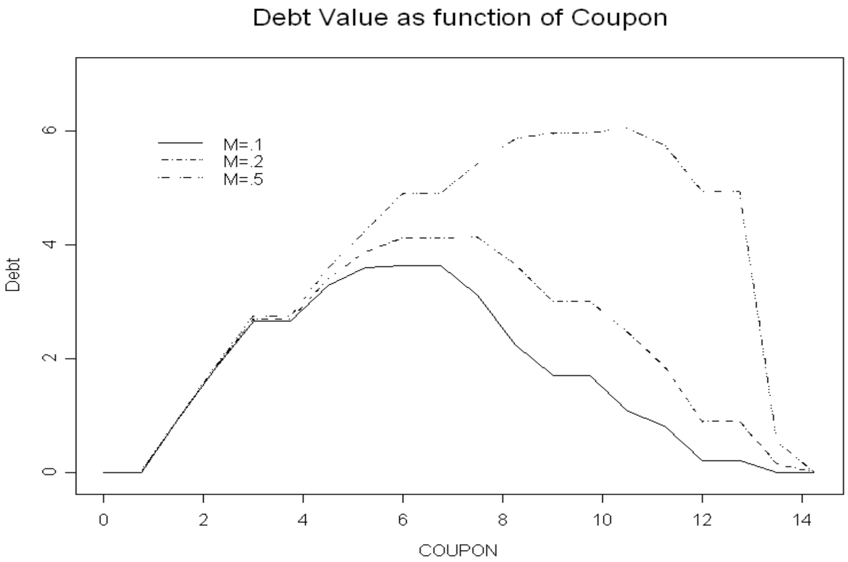
- v current value of firm's assets
- p_0 option premium
- M length of the option life
- κ strike of the option
- c_t cost of servicing the existing debt


Expected Bankruptcy Time as function of Coupon
Tau
0.0
0.5
1.0
1.5
2.0
0
2
4
6
8
10
12
14
COUPON
M=.1
M=.2
M=.5

Plant Value


Plant Value as function of Coupon
PlantValue
0
500
1000
1500
2000
2500
COUPON
0
2
4
6
8
10
12
14
M=.1
M=.2
M=.5

Debt Value


Debt Value as function of Coupon
M=.1
M=.2
M=.5
Debt
0
2
4
6
0
2
4
6
8
10
12
14
COUPON

Pricing Calendar Spreads in Forward Models

Involves prices of two forward contracts with different maturities, say T_1 and T_2

$$S_1(t) = F(t, T_1) \qquad \text{and} \qquad S_2(t) = F(t, T_2),$$

Remember forward prices **are** log-normal

Price at time t of a calendar spread option with maturity T and strike K

$$\alpha = e^{-r[T-t]} F(t, T_2), \quad \beta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_k(s, T_2)^2 ds},$$

$$\gamma = e^{-r[T-t]} F(t, T_1), \quad \text{and} \qquad \delta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_k(s, T_1)^2 ds}$$

and $\kappa = e^{-r(T-t)}$ ($\mu \equiv 0$ per risk-neutral dynamics)

$$\rho = \frac{1}{\beta\delta} \sum_{k=1}^{n} \int_t^T \sigma_k(s, T_1) \sigma_k(s, T_2)\, ds$$

Pricing Spark Spreads in Forward Models

Cross-commodity

- subscript **e** for forward prices, times-to-maturity, volatility functions, ... relative to electric power
- subscript **g** for quantities pertaining to natural gas.

Pay-off

$$\left(F_e(T, T_e) - H * F_g(T, T_g) - K\right)^+ .$$

- $T < \min\{T_e, T_g\}$
- Heat rate H
- Strike K given by O& M costs

Natural

- **Buyer** owner of a power plant that transforms gas into electricity,
- **Protection** against low electricity prices and/or high gas prices.

Joint Dynamics of the Commodities

$$\begin{cases} dF_e(t,T_e) &= F_e(t,T_e)[\mu_e(t,T_e)dt + \sum_{k=1}^n \sigma_{e,k}(t,T_e)dW_k(t)] \\ dF_g(t,T_g) &= F_g(t,T_g)[\mu_g(t,T_g)dt + \sum_{k=1}^n \sigma_{g,k}(t,T_g)dW_k(t)] \end{cases}$$

- Each commodity has its own volatility factors
- between The two dynamics share the **same** driving Brownian motion processes W_k, hence **correlation.**

Fitting Join Cross-Commodity Models

- on any given day t we have
  - electricity forward contract prices for $N^{(e)}$ times-to-maturity $\tau_1^{(e)} < \tau_2^{(e)}, \ldots < \tau_{N^{(e)}}^{(e)}$
  - natural gas forward contract prices for $N^{(g)}$ times-to-maturity $\tau_1^{(g)} < \tau_2^{(g)}, \ldots < \tau_{N^{(g)}}^{(g)}$

  Typically $N^{(e)} = 12$ and $N^{(g)} = 36$ (possibly more).
  - Estimate instantaneous vols $\sigma^{(e)}(t)$ & $\sigma^{(g)}(t)$ 30 days rolling window
  - For each day t, the $N = N^{(e)} + N^{(g)}$ dimensional random vector $\mathbf{X}(t)$

$$\mathbf{X}(t) = \left[\begin{array}{l} \left(\frac{\log \tilde{F}_e(t+1,\tau_j^{(e)}) - \log \tilde{F}_e(t,\tau_j^{(e)})}{\sigma^{(e)}(t)} \right)_{j=1,\ldots,N^{(e)}} \\ \left(\frac{\log \tilde{F}_g(t+1,\tau_j^{(g)}) - \log \tilde{F}_g(t,\tau_j^{(g)})}{\sigma^{(g)}(t)} \right)_{j=1,\ldots,N^{(g)}} \end{array} \right]$$

  - Run PCA on historical samples of $\mathbf{X}(t)$
  - Choose small number n of factors
  - for $k = 1, \ldots, n$,
    - first $N^{(e)}$ coordinates give the electricity volatilities $\tau \hookrightarrow \sigma_k^{(e)}(\tau)$ for $k = 1, \ldots, n$
    - remaining $N^{(g)}$ coordinates give the gas volatilities $\tau \hookrightarrow \sigma_k^{(g)}(\tau)$.

Skip gory details

Price at time t

$$p_t = e^{-r(T-t)}\mathbb{E}_t\left\{(F_e(T, T_e) - H * F_g(T, T_g) - K)^+\right\}$$

$F_e(T, T_e)$ and $F_g(T, T_g)$ are log-normal under the pricing measure calibrated by PCA

$$F_e(T, T_e) = F_e(t, T_e)\exp\left[-\frac{1}{2}\sum_{k=1}^{n}\int_t^T \sigma_{e,k}(s, T_e)^2 ds + \sum_{k=1}^{n}\int_t^T \sigma_{e,k}(s, T_e)dW_k(s)\right]$$

and:

$$F_g(T, T_g) = F_g(t, T_g)\exp\left[-\frac{1}{2}\sum_{k=1}^{n}\int_t^T \sigma_{g,k}(s, T_g)^2 ds + \sum_{k=1}^{n}\int_t^T \sigma_{g,k}(s, T_g)dW_k(s)\right]$$

Set

$$S_1(t) = H * F_g(t, T_g) \qquad \text{and} \qquad S_2(t) = F_e(t, T_e)$$

Use the constants

$$\alpha = e^{-r(T-t)} F_e(t, T_e), \qquad \text{and} \qquad \beta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_{e,k}(s, T_e)^2 \, ds}$$

for the first log-normal distribution,

$$\gamma = He^{-r(T-t)} F_g(t, T_g), \qquad \text{and} \qquad \delta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_{g,k}(s, T_g)^2 \, ds}$$

for the second one, $\kappa = e^{-r(T-t)} K$ and

$$\rho = \frac{1}{\beta\delta} \int_t^T \sum_{k=1}^{n} \sigma_{e,k}(s, T_e) \sigma_{g,k}(s, T_g) ds$$

for the correlation coefficient.

- Fourier Approximations (Madan, Carr, Dempster, . . .)
- Bachelier approximation
- Zero-strike approximation
- Kirk approximation
- Upper and Lower Bounds

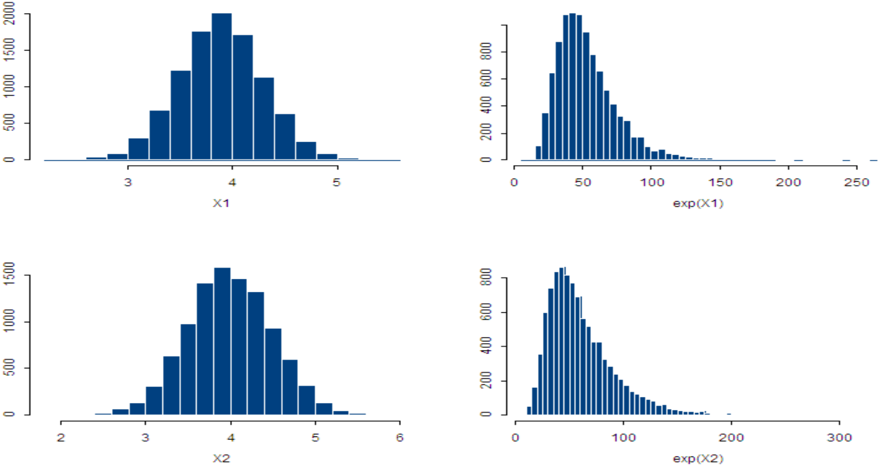
Can we also approximate the **Greeks** ?

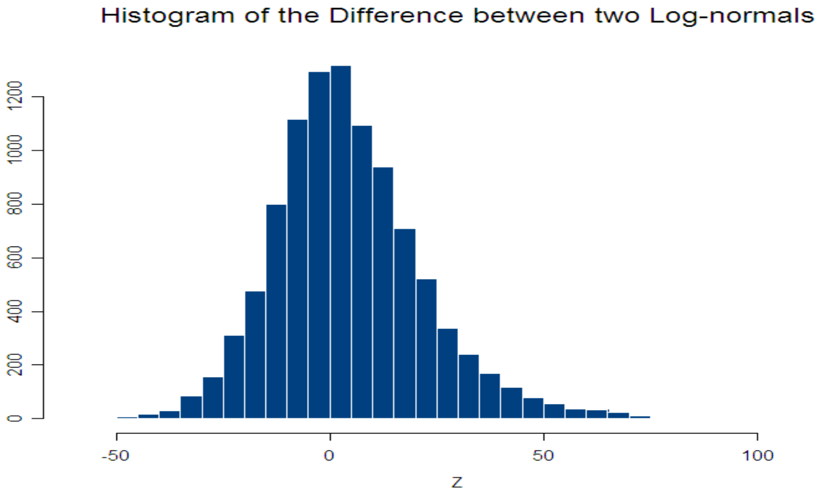
Bachelier Approximation

- Generate $x_1^{(1)}, x_2^{(1)}, \cdots, x_N^{(1)}$ from $N(\mu_1, \sigma_1^2)$
- Generate $x_1^{(2)}, x_2^{(2)}, \cdots, x_N^{(2)}$ from $N(\mu_1, \sigma_1^2)$
- Correlation ρ
- Look at the distribution of

$$e^{x_1^{(2)}} - e^{x_1^{(1)}}, e^{x_2^{(2)}} - e^{x_2^{(1)}}, \cdots, e^{x_N^{(2)}} - e^{x_N^{(1)}}$$

Log-Normal Samples




Histogram of the Difference between two Log-normals
0
200
400
600
800
1000
1200
-50
0
50
100
Z

Bachelier Approximation

- Assume $(S_2(T) - S_1(T)$ is Gaussian
- Match the first two moments

$$p = \left(m(T) - Ke^{-rT}\right)\Phi\left(\frac{m(T) - Ke^{-rT}}{s(T)}\right) + s(T)\varphi\left(\frac{m(T) - Ke^{-rT}}{s(T)}\right)$$

with:

$$\begin{aligned} m(T) &= (x_2 - x_1)e^{(\mu - r)T} \\ s^2(T) &= e^{2(\mu - r)T}\left[x_1^2\left(e^{\sigma_1^2 T} - 1\right) - 2x_1x_2\left(e^{\rho\sigma_1\sigma_2 T} - 1\right) + x_2^2\left(e^{\sigma_2^2 T} - 1\right)\right] \end{aligned}$$

Easy to compute the Greeks !

Zero-Strike Approximation

$$p = e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\}$$

- Assume $S_2(T) = F_E(T)$ is **log-normal**
- Replace $S_1(T) = H * F_G(T)$ by $\tilde{S}_1(T) = S_1(T) + K$
- Assume $S_2(T)$ and $\tilde{S}_1(T)$ are **jointly log-normal**
- Use **Margrabe** formula for $p = e^{-rT}\mathbb{E}\{(S_2(T) - \tilde{S}_1(T))^+\}$

Use the Greeks from Margrabe formula !

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Use the Greeks from Margrabe formula !

$$\hat{p}^K = x_2 \Phi\left(\frac{\ln\left(\frac{x_2}{x_1 + Ke^{-rT}}\right)}{\sigma^K} + \frac{\sigma^K}{2}\right) - (x_1 + Ke^{-rT})\Phi\left(\frac{\ln\left(\frac{x_2}{x_1 + Ke^{-rT}}\right)}{\sigma^K} - \frac{\sigma^K}{2}\right)$$

where

$$\sigma^K = \sqrt{\sigma_2^2 - 2\rho\sigma_1\sigma_2 \frac{x_1}{x_1 + Ke^{-rT}} + \sigma_1^2 \left(\frac{x_1}{x_1 + Ke^{-rT}}\right)^2}.$$

Exactly what we called ”Zero Strike Approximation”!!!

Upper and Lower Bounds

$$\Pi(\alpha, \beta, \gamma, \delta, \kappa, \rho) = \mathbb{E}\left\{\left(\alpha e^{\beta X_1 - \beta^2/2} - \gamma e^{\delta X_2 - \delta^2/2} - \kappa\right)^+\right\}$$

where

- α, β, γ, δ and κ real constants
- X_1 and X_2 are jointly Gaussian $N(0, 1)$
- correlation ρ

$\alpha = x_2 e^{-q_2 T} \quad \beta = \sigma_2 \sqrt{T} \quad \gamma = x_1 e^{-q_1 T} \quad \delta = \sigma_1 \sqrt{T}$ and $\kappa = K e^{-rT}$.

A Precise Lower Bound

$$\hat{p} = x_2 e^{-q_2 T} \Phi\left(d^* + \sigma_2 \cos(\theta^* + \phi)\sqrt{T}\right) - x_1 e^{-q_1 T} \Phi\left(d^* + \sigma_1 \sin\theta^* \sqrt{T}\right) - K e^{-rT} \Phi(d^*)$$

where

- θ^* is the solution of

$$\frac{1}{\delta\cos\theta} \ln\left(-\frac{\beta\kappa\sin(\theta+\phi)}{\gamma[\beta\sin(\theta+\phi) - \delta\sin\theta]}\right) - \frac{\delta\cos\theta}{2} = \frac{1}{\beta\cos(\theta+\phi)} \ln\left(-\frac{\delta\kappa\sin\theta}{\alpha[\beta\sin(\theta+\phi) - \delta\sin\theta]}\right) - \frac{\beta\cos(\theta+\phi)}{2}$$

- the angle ϕ is defined by setting $\rho = \cos\phi$
- d^* is defined by

$$d^* = \frac{1}{\sigma\cos(\theta^* - \psi)\sqrt{T}} \ln\left(\frac{x_2 e^{-q_2 T}\sigma_2 \sin(\theta^* + \phi)}{x_1 e^{-q_1 T}\sigma_1 \sin\theta^*}\right) - \frac{1}{2}(\sigma_2\cos(\theta^* + \phi) + \sigma_1\cos\theta$$

- the angles ϕ and ψ are chosen in $[0, \pi]$ such that:

$$\cos\phi = \rho \qquad \text{and} \qquad \cos\psi = \frac{\sigma_1 - \rho\sigma_2}{\sigma},$$

Remarks on this Lower Bound

- $\hat{p}$ is equal to the true price p when
  - $K = 0$
  - $x_1 = 0$
  - $x_2 = 0$
  - $\rho = -1$
  - $\rho = +1$
- Margrabe formula when $K = 0$ because

$$\theta^* = \pi + \psi = \pi + \arccos\left(\frac{\sigma_1 - \rho\sigma_2}{\sigma}\right).$$

with:

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

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with:

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

Delta Hedging

The portfolio comprising at each time $t \leq T$

$$\Delta_1 = -e^{-q_1 T}\Phi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right)$$

and

$$\Delta_2 = e^{-q_2 T}\Phi\left(d^* + \sigma_2 \cos(\theta^* + \phi)\sqrt{T}\right)$$

units of each of the underlying assets is a **sub-hedge**

its value at maturity is a.s. a ***lower bound*** *for the pay-off*

The Other Greeks

- ϑ_1 and ϑ_2 sensitivities w.r.t. volatilities σ_1 and σ_2
- χ sensitivity w.r.t. correlation ρ
- κ sensitivity w.r.t. strike price K
- Θ sensitivity w.r.t. maturity time T

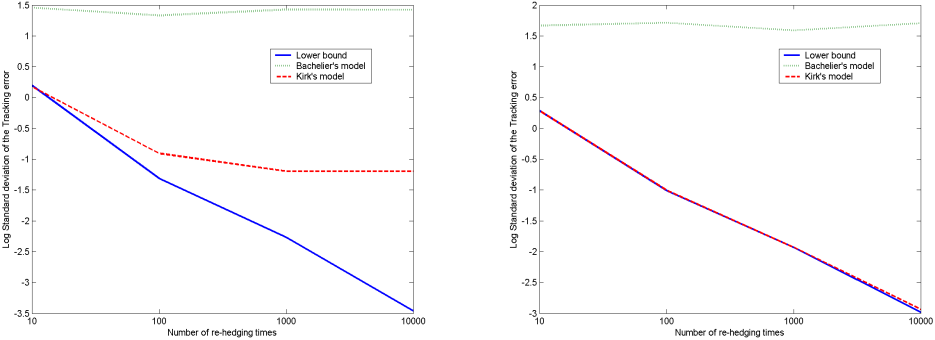
$$
\begin{aligned}
\vartheta_1 &= x_1 e^{-q_1 T} \varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right) \cos\theta^* \sqrt{T} \\
\vartheta_2 &= -x_2 e^{-q_2 T} \varphi\left(d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T}\right) \cos(\theta^* + \phi) \sqrt{T} \\
\chi &= -x_1 e^{-q_1 T} \varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right) \sigma_1 \frac{\sin\theta^*}{\sin\phi} \sqrt{T} \\
\kappa &= -\Phi\left(d^*\right) e^{-rT} \\
\Theta &= \frac{\sigma_1 \vartheta_1 + \sigma_2 \vartheta_2}{2T} - q_1 x_1 \Delta_1 - q_2 x_2 \Delta_2 - rK\kappa
\end{aligned}
$$

The Other Greeks

- ϑ_1 and ϑ_2 sensitivities w.r.t. volatilities σ_1 and σ_2
- χ sensitivity w.r.t. correlation ρ
- κ sensitivity w.r.t. strike price K
- Θ sensitivity w.r.t. maturity time T

$$
\begin{aligned}
\vartheta_1 &= x_1 e^{-q_1 T} \varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right) \cos\theta^* \sqrt{T} \\
\vartheta_2 &= -x_2 e^{-q_2 T} \varphi\left(d^* + \sigma_2 \cos(\theta^* + \phi) \sqrt{T}\right) \cos(\theta^* + \phi) \sqrt{T} \\
\chi &= -x_1 e^{-q_1 T} \varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right) \sigma_1 \frac{\sin\theta^*}{\sin\phi} \sqrt{T} \\
\kappa &= -\Phi\left(d^*\right) e^{-rT} \\
\Theta &= \frac{\sigma_1\vartheta_1 + \sigma_2\vartheta_2}{2T} - q_1 x_1 \Delta_1 - q_2 x_2 \Delta_2 - rK\kappa
\end{aligned}
$$

Comparisons



Behavior of the tracking error as the number of re-hedging times increases. The model data are $x_1 = 100$, $x_2 = 110$, $\sigma_1 = 10\%$, $\sigma_2 = 15\%$ and $T = 1$. $\rho = 0.9$, $K = 30$ (left) and $\rho = 0.6$, $K = 20$ (right).

Generalization: European Basket Option

Black-Scholes Set-Up

- Multidimensional model
- n stocks $S_1, \ldots, S_n$
- Risk neutral dynamics

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t),$$

- initial values $S_1(0), \ldots, S_n(0)$
- $B_1, \ldots, B_n$ independent standard Brownian motions
- Correlation through matrix (σ_{ij})

European Basket Option (cont.)

- Vector of weights $(w_i)_{i=1,\dots,n}$ (most often $w_i \geq 0$)
- Basket option struck at K at maturity T given by payoff

$$\left(\sum_{i=1}^{n} w_i S_i(T) - K \right)^{+}$$

(**Asian Options**)

Risk neutral valuation: price at time 0

$$p = e^{-rT} \mathbb{E} \left\{ \left(\sum_{i=1}^{n} w_i S_i(T) - K \right)^{+} \right\}$$

Existing Literature

- **Jarrow** and **Rudd**
  - Replace true distribution by simpler distribution with same first moments
  - Edgeworth (Charlier) expansions
  - Bachelier approximation when Gaussian distribution used
- **SemiParametric** Bounds (known marginals)
- Fully **NonParametric** Bounds
  - Intervals too large
  - Used only to rule out arbitrage
- Replacing Arithmetic Averages by Geometric Averages (**Musiela**)

Reformulation of the Problem

- Change w_i if necessary to absorb exponent mean
- Change w_i if necessary to introduce variance in exponent
- Replace K by $-w_0 e^{G_0 - \text{var}\{G_0\}/2}$ with $G_0 \sim N(0,0)$
- Set $x_i = |w_i|$ and $\epsilon_i = \text{sign}(w_i)$

Our original problem becomes: **Compute**

$$\mathbb{E}\{X^+\}$$

for

$$X = \sum_{i=0}^{n} \epsilon_i x_i e^{G_i - \text{Var}(G_i)/2}.$$

What Are We Looking For?

- Explicit formulae in close form
- Compute Greeks as well

$n = 1$

- **Black Scholes** Formula
- **Margrabe** Formula

Two Optimization Problems

For any $X \in L^1$,

$$\sup_{0 \le Y \le 1} \mathbb{E}\{XY\} = \mathbb{E}\{X^+\} = \inf_{X = Z_1 - Z_2, Z_1 \ge 0, Z_2 \ge 0} \mathbb{E}\{Z_1\}.$$

Lower Bound Strategy

$$\sup_{0 \le Y \le 1} \mathbb{E}\{XY\} = \mathbb{E}\{X^+\}$$

- Compute sup in LHS restricting Y
- We choose $Y = \mathbf{1}_{\{u \cdot G \le d\}}$ for $u \in \mathbb{R}^{n+1}$ and $d \in \mathbb{R}$

where $G = (G_0, G_1, \ldots, G_n)$ and $u \cdot G = u_0 G_0 + u_1 G_1 + \ldots + u_n G_n$

Can we compute?

$$p_* = \sup_{u,d} \mathbb{E}\left\{X \mathbf{1}_{\{u \cdot G \le d\}}\right\}$$

We sure can!

$$\mathbb{E}\left\{X \mathbf{1}_{\{u \cdot G \le d\}}\right\} = \sum_{i=0}^{n} \mathbb{E}\left\{\epsilon_i x_i \mathbb{E}\{e^{G_i - \mathrm{Var}(G_i)/2} | u \cdot G\} \mathbf{1}_{\{u \cdot G \le d\}}\right\}$$

Lower Bound

$$
\begin{aligned}
p_* &= \sup_{d\in\mathbb{R}} \sup_{u\cdot\Sigma u=1} \sum_{i=0}^{n} \epsilon_i x_i \Phi\left(d + (\Sigma u)_i\right) \\
&= \sup_{d\in\mathbb{R}} \sup_{\|v\|=1} \sum_{i=0}^{n} \epsilon_i x_i \Phi\left(d + \sigma_i(\sqrt{C}v)_i\right).
\end{aligned}
$$

where

$$C = D\Sigma D \qquad \text{and} \qquad D = \mathrm{diag}(1/\sigma_i)$$

and

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \text{and} \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

First Order Conditions

Lagrangian $\mathcal{L}$:

$$\mathcal{L}(v,d) = \sum_{i=0}^{n} \epsilon_i x_i \Phi\left(d + \sigma_i(\sqrt{C}v)_i\right) - \frac{\mu}{2}\left(\|v\|^2 - 1\right).$$

$$p_* = \sum_{i=0}^{n} \epsilon_i x_i \Phi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right)$$

where d^* and v^* satisfy the following first order conditions

$$\begin{aligned}
\sum_{i=0}^{n} \epsilon_i x_i \sigma_i \sqrt{C}_{ij} \varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right) - \mu v_j^* &= 0 \quad \text{for } j = 0, \ldots, n \\
\sum_{i=0}^{n} \epsilon_i x_i \varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right) &= 0 \\
\|v^*\| &= 1.
\end{aligned}$$

Remark (Warm Up for Upper Bound)

for each k in $\{0, 1, \ldots, n\}$

$$
\begin{aligned}
X &= \sum_{i \neq k} \varepsilon_i x_i e^{G_i - \mathrm{Var}(G_i)/2} - \lambda_i^k x_k e^{G_k - \mathrm{Var}(G_k)/2} \\
&= \sum_{i \neq k} \left(\varepsilon_i x_i e^{G_i - \mathrm{Var}(G_i)/2} - \lambda_i^k x_k e^{G_k - \mathrm{Var}(G_k)/2} \right)^+ \\
&\quad - \sum_{i \neq k} \left(\varepsilon_i x_i e^{G_i - \mathrm{Var}(G_i)/2} - \lambda_i^k x_k e^{G_k - \mathrm{Var}(G_k)/2} \right)^-
\end{aligned}
$$

if $\sum_{i \neq k} \lambda_i^k = -\varepsilon_k$

Upper Bound Strategy

In formula

$$\mathbb{E}\{X^+\} = \inf_{X=Z_1-Z_2, Z_1\geq 0, Z_2 \geq 0} \mathbb{E}\{Z_1\}.$$

Restrict Z_1 to

$$\sum_{i\neq k}\left(\varepsilon_i x_i e^{G_i - \mathrm{Var}(G_i)/2} - \lambda_i^k \tilde{x}_k e^{G_k - \mathrm{Var}(G_k)/2}\right)^+$$

where $k = 0, \dots, n$, $\sum_{i\neq k} \lambda_i^k = -\varepsilon_k$ and $\lambda_i^k \varepsilon_i > 0$ for all $i \neq k$.

Upper Bound

$$p^* = \min_{0 \le k \le n} \left\{ \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d^k + \varepsilon_i \sigma_i^k\right) \right\}$$

where d^k is given by the following first order conditions

$$\begin{aligned} \frac{\varepsilon_i}{\sigma_i^k} \ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k x_k}\right) - \frac{\varepsilon_i \sigma_i^k}{2} &= \frac{\varepsilon_j}{\sigma_j^k} \ln\left(\frac{\varepsilon_j x_j}{\lambda_j^k x_k}\right) - \frac{\varepsilon_j \sigma_j^k}{2} = d^k \quad \text{for } i, j \neq k \\ \sum_{i \neq k} \lambda_i^k &= -\tilde{\varepsilon}_k \\ \lambda_i^k \varepsilon_i &> 0 \quad \text{for } i \neq k. \end{aligned}$$

Equality between Bounds

If for all $i, j = 0, \ldots, n$,

$$\Sigma_{ij} = \varepsilon_i \varepsilon_j \sigma_i \sigma_j,$$

then

$$p_* = p^*.$$

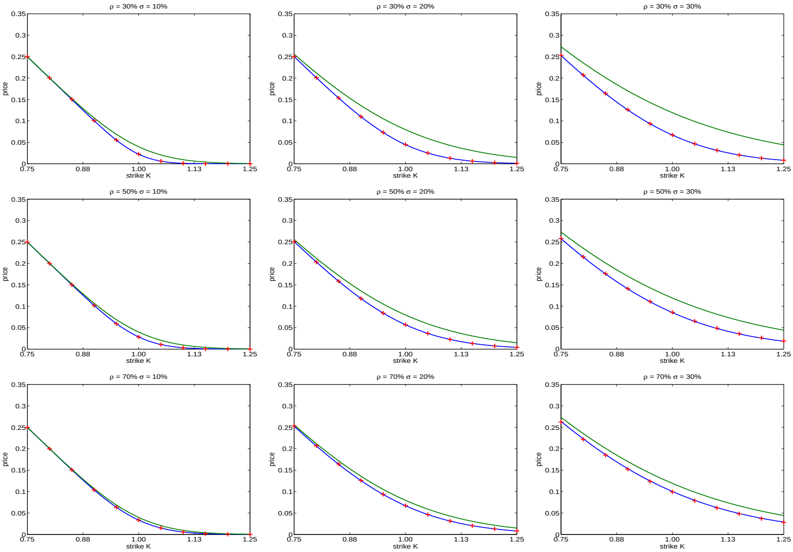
Error Bound

$$0 \leq p^* - p_* \leq \sqrt{\frac{2}{\pi}} \min_{0 \leq k \leq n} \left\{ \sum_{i=0}^{n} x_i \sigma_i^k \right\}.$$

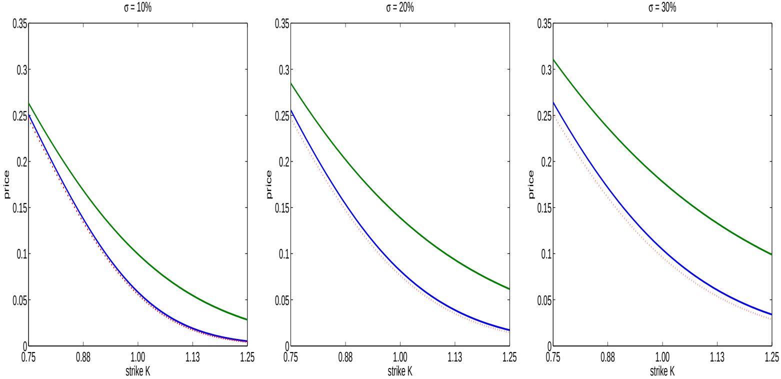
where

$$\sigma_i^k = \sqrt{\mathrm{Var}\left(\{G_i - G_k\}\right)}$$

Numerical Performance



Asian Options



Lower and upper bound on the price of an Asian option.
The dotted line represents the geometric average approximation.

Computation of (Approximate) Greeks

$$
\begin{aligned}
\Delta_{*i} = \frac{\partial p_*}{\partial x_i} &= \varepsilon_i \Phi \left(d^* + \sigma_i (\sqrt{C} v^*)_i \right) \\
Vega_{*i} = \frac{\partial p_*}{\partial \sigma_i} \sqrt{T} &= \varepsilon_i x_i (\sqrt{C} v^*)_i \varphi \left(d^* + \sigma_i (\sqrt{C} v^*)_i \right) \sqrt{T} \\
\chi_{*ij} = \frac{\partial p_*}{\partial \rho_{ij}} &= \frac{1}{2} \sum_{k=0}^{n} \varepsilon_k x_k \left(\sigma_i C_{kj}^{-\frac{1}{2}} v_j^* + \sigma_j C_{ki}^{-\frac{1}{2}} v_i^* \right) \varphi \left(d^* + \sigma_k (\sqrt{C} v^*) \right. \\
\Theta_* = \frac{\partial p_*}{\partial T} &= \frac{1}{2T} \sum_{k=0}^{n} \varepsilon_k x_k \sigma_k (\sqrt{C} v^*)_k \varphi \left(d^* + \sigma_k (\sqrt{C} v^*)_k \right).
\end{aligned}
$$

Second Order Derivatives

$$\Gamma_{*ij} = \varepsilon_i \varepsilon_j \frac{\varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\right)\varphi\left(d^* + \sigma_j(\sqrt{C}v^*)_j\right)}{\sum_{k=0}^n \varepsilon_k x_k \sigma_k(\sqrt{C}v^*)_k \varphi\left(d^* + \sigma_k(\sqrt{C}v^*)_k\right)},$$

then

$$-\Theta_* + \frac{1}{2T}\sum_{i=0}^n \sum_{j=0}^n \Sigma_{ij} x_i x_j \Gamma_{*ij} = 0.$$

Option Payoff

$$\left(\sum_{i=1}^{n} w_i S_i(T) - K\right)^{+} \mathbf{1}_{\{\inf_{t\le T} S_1(t)\ge H\}}.$$

Option price is

$$\mathbb{E}\left\{\left(\sum_{i=0}^{n} \varepsilon_i x_i e^{G_i(1)-\frac{1}{2}\sigma_i^2} \mathbf{1}_{\left\{\inf_{\theta\le 1} x_1 e^{G_1(\theta)-\frac{1}{2}\sigma_1^2\theta}\ge H\right\}}\right)^{+}\right\},$$

where

- $\varepsilon_1 = +1$, $\sigma_1 > 0$ and $H < x_1$
- $\{G(\theta); \theta \le 1\}$ is a $(n+1)$-dimensional Brownian motion starting from 0 with covariance Σ.

Use lower bound.

$$p_* = \sup_{d,u} \mathbb{E} \left\{ \sum_{i=0}^{n} \varepsilon_i x_i e^{G_i(1) - \frac{1}{2}\sigma_i^2} \mathbf{1}_{\left\{ \inf_{\theta \leq 1} x_1 e^{G_1(\theta) - \frac{1}{2}\sigma_1^2 \theta} \geq H; u \cdot G(1) \leq d \right\}} \right\}.$$

Girsanov implies

$$p_* = \sup_{d,u} \sum_{i=0}^{n} \varepsilon_i x_i \mathbb{P} \left\{ \inf_{\theta \leq 1} G_1(\theta) \right.$$

$$\left. + \left(\Sigma_{i1} - \sigma_1^2/2 \right) \theta \geq \ln\left(\frac{H}{x_1} \right); u \cdot G(1) \leq d - (\Sigma u)_i \right\}.$$

Numerical Results

σ	ρ	H/x_1	$n = 10$	$n = 20$	$n = 30$
0.4	0.5	0.7	0.1006	0.0938	0.0939
0.4	0.5	0.8	0.0811	0.0785	0.0777
0.4	0.5	0.9	0.0473	0.0455	0.0449
0.4	0.7	0.7	0.1191	0.1168	0.1165
0.4	0.7	0.8	0.1000	0.1006	0.0995
0.4	0.7	0.9	0.0608	0.0597	0.0594
0.4	0.9	0.7	0.1292	0.1291	0.1290
0.4	0.9	0.8	0.1179	0.1175	0.1173
0.4	0.9	0.9	0.0751	0.0747	0.0745
0.5	0.5	0.7	0.1154	0.1122	0.1110
0.5	0.5	0.8	0.0875	0.0844	0.0816
0.5	0.5	0.9	0.0518	0.0464	0.0458
0.5	0.7	0.7	0.1396	0.1389	0.1388
0.5	0.7	0.8	0.1103	0.1086	0.1080
0.5	0.7	0.9	0.0631	0.0619	0.0615
0.5	0.9	0.7	0.1597	0.1593	0.1592
0.5	0.9	0.8	0.1328	0.1322	0.1320
0.5	0.9	0.9	0.0786	0.0782	0.0780

Valuing a Tolling Agreement

Stylized Version

- **Leasing an Energy Asset**
  - Fossil Fuel Power Plant
  - Oil Refinery
  - Pipeline
- **Owner of the Agreement**
  - Decides **when** and **how** to use the asset (e.g. run the power plant)
  - Has someone else do the leg work

Plant Operation Model: the Finite Mode Case

- Markov process (**state of the world**) $X_t = (X_t^{(1)}, X_t^{(2)}, \cdots)$ (e.g. $X_t^{(1)} = P_t, \quad X_t^{(2)} = G_t, \quad X_t^{(3)} = O_t$ for a dual plant)
- Plant **characteristics**
  - $\mathbb{Z}_M \triangleq \{0, \cdots, M-1\}$ **modes** of operation of the plant
  - $H_0, H_1 \cdots, H_{M-1}$ **heat rates**
  - $\{C(i,j)\}_{(i,j)\in\mathbb{Z}_M}$ regime **switching costs** $(C(i,j) = C(i,\ell) + C(\ell,j))$
  - $\psi_i(t,x)$ **reward** at time t when world in state x, plant in mode i
- **Operation** of the plant (control) $u = (\xi, \mathcal{T})$ where
  - $\xi_k \in \mathbb{Z}_M \triangleq \{0, \cdots, M-1\}$ successive modes
  - $0 \leqslant \tau_{k-1} \leqslant \tau_k \leqslant T$ switching times

Plant Operation Model: the Finite Mode Case

- T (**horizon**) length of the tolling agreement
- Total **reward**

$$H(x, i, [0, T]; u)(\omega) \triangleq \int_0^T \psi_{u_s}(s, X_s)\, ds - \sum_{\tau_k < T} C(u_{\tau_k -}, u_{\tau_k})$$

- $\mathcal{U}(t)$) *acceptable* controls on $[t, T]$
  (adapted càdlàg $\mathbb{Z}_M$-valued processes u of a.s. finite variation on $[t, T]$)

Optimal Switching Problem

$$J(t, x, i) = \sup_{u \in \mathcal{U}(t)} J(t, x, i; u),$$

where

$$\begin{aligned} J(t, x, i; u) &= \mathbb{E}\big[H(x, i, [t, T]; u) \big| X_t = x, u_t = i\big] \\ &= \mathbb{E}\big[\int_0^T \psi_{u_s}(s, X_s)\, ds - \sum_{\tau_k < T} C(u_{\tau_k -}, u_{\tau_k}) \big| X_t = x, u_t = i\big] \end{aligned}$$

Iterative Optimal Stopping

Consider problem with **at most** k mode switches

$$\mathcal{U}^k(t) \triangleq \{(\xi, \mathcal{T}) \in \mathcal{U}(t) \colon \tau_\ell = T \text{ for } \ell \geqslant k+1\}$$

Admissible strategies on $[t, T]$ with at most k switches

$$J^k(t,x,i) \triangleq \operatorname{esssup}_{u \in \mathcal{U}^k(t)} \mathbb{E}\Big[\int_t^T \psi_{u_s}(s, X_s)\, ds - \sum_{t \leqslant \tau_k < T} C(u_{\tau_k -}, u_{\tau_k}) \Big| X_t = x, u_t = i\Big].$$

Alternative Recursive Construction

$$J^0(t,x,i) \triangleq \mathbb{E}\Big[\int_t^T \psi_i(s,X_s)\,ds \Big| X_t = x\Big],$$

$$J^k(t,x,i) \triangleq \sup_{\tau\in\mathcal{S}_t} \mathbb{E}\Big[\int_t^\tau \psi_i(s,X_s)\,ds + \mathcal{M}^{k,i}(\tau,X_\tau)\Big| X_t = x\Big].$$

Intervention operator $\mathcal{M}$

$$\mathcal{M}^{k,i}(t,x) \triangleq \max_{j\neq i}\Big\{-C_{i,j} + J^{k-1}(t,x,j)\Big\}.$$

Hamadène - Jeanblanc (M=2)

Variational Formulation

Notation

- $\mathcal{L}_X$ X space-time generator of Markov process X_t in $\mathbb{R}^d$
- $\mathcal{M}\phi(t,x,i) = \max_{j\neq i}\{-C_{i,j} + \phi(t,x,j)\}$ intervention operator

Assume

- $\phi(t,x,i)$ in $\mathcal{C}^{1,2}\big(([0,T]\times\mathbb{R}^d)\setminus D\big)\cap\mathcal{C}^{1,1}(D)$
- $D = \cup_i\big\{(t,x)\colon \phi(t,x,i) = \mathcal{M}\phi(t,x,i)\big\}$
- (**QVI**) for all $i\in\mathbb{Z}_M$:
  1. $\phi \geqslant \mathcal{M}\phi$,
  2. $\mathbb{E}^x\left[\int_0^T \nVdash_{\phi\leqslant\mathcal{M}\phi}\, dt\right] = 0$,
  3. $\mathcal{L}_X\phi(t,x,i) + \psi_i(t,x) \leqslant 0, \qquad \phi(T,x,i) = 0$,
  4. $\Big(\mathcal{L}_X\phi(t,x,i) + \psi_i(t,x)\Big)\Big(\phi(t,x,i) - \mathcal{M}\phi(t,x,i)\Big) = 0.$

Conclusion

ϕ is the optimal value function for the switching problem

Reflected Backward SDE's

Assume

- $X_0 = x$ & $\exists(Y^x, Z^x, A)$ adapted to $(\mathcal{F}_t^X)$

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |Y_t^x|^2 + \int_0^T \|Z_t^x\|^2\, dt + |A_T|^2\Big] < \infty$$

and

$$Y_t^x = \int_t^T \psi_i(s, X_s^x)\, ds + A_T - A_t - \int_t^T Z_s \cdot dW_s,$$
$$Y_t^x \geqslant \mathcal{M}^{k,i}(t, X_t^x),$$
$$\int_0^T (Y_t^x - \mathcal{M}^{k,i}(t, X_t^x))\, dA_t = 0, \qquad A_0 = 0.$$

Conclusion: if $Y_0^x = J^k(0, x, i)$ then

$$Y_t^x = J^k(t, X_t^x, i)$$

System of Reflected Backward SDE's

QVI for optimal switching: **coupled system** of reflected BSDE's for $(Y^i)_{i \in \mathbb{Z}_M}$,

$$Y^i_t = \int_t^T \psi_i(s, X_s)\, ds + A^i_T - A^i_t - \int_t^T Z^i_s \cdot dW_s,$$
$$Y^i_t \geqslant \max_{j \neq i} \{-C_{i,j} + Y^j_t\}.$$

Existence and uniqueness Directly for $M > 2$?
$M = 2$, **Hamadène - Jeanblanc** use difference process $Y^1 - Y^2$.

- Time Step $\Delta t = T/M^\sharp$
- Time grid $\mathcal{S}^\Delta = \{m\Delta t,\, m = 0, 1, \dots, M^\sharp\}$
- Switches are allowed in $\mathcal{S}^\Delta$

DPP

For $t_1 = m\Delta t, t_2 = (m+1)\Delta t$ consecutive times

$$J^k(t_1, X_{t_1}, i) = \max\Big(\mathbb{E}\Big[\int_{t_1}^{t_2} \psi_i(s, X_s)\, ds + J^k(t_2, X_{t_2}, i) | \mathcal{F}_{t_1}\Big],\, \mathcal{M}^{k,i}(t_1, X_{t_1})\Big)$$
$$\simeq \Big(\psi_i(t_1, X_{t_1})\,\Delta t + \mathbb{E}\big[J^k(t_2, X_{t_2}, i) | \mathcal{F}_{t_1}\big]\Big) \vee \Big(\max_{j \neq i}\{-C_{i,j} + J^{k-1}(t_1, X_{t_1}, j)\}\Big). \tag{1}$$

Tsitsiklis - van Roy

Longstaff-Schwartz Version

Recall

$$J^k(m\Delta t, x, i) = \mathbb{E}\Big[\sum_{j=m}^{\tau^k} \psi_i(j\Delta t, X_{j\Delta t})\,\Delta t + \mathcal{M}^{k,i}(\tau^k\Delta t, X_{\tau^k\Delta t})\Big|\, X_{m\Delta t} = x\Big].$$

Analogue for τ^k:

$$\tau^k(m\Delta t, x^\ell_{m\Delta t}, i) = \begin{cases} \tau^k((m+1)\Delta t, x^\ell_{(m+1)\Delta t}, i), & \text{no switch;} \\ m, & \text{switch,} \end{cases} \qquad (2)$$

and the set of paths on which we switch is given by $\{\ell \colon \hat{\jmath}^\ell(m\Delta t; i) \neq i\}$ with

$$\hat{\jmath}^\ell(t_1; i) = \arg\max_j \Big(-C_{i,j} + J^{k-1}(t_1, x^\ell_{t_1}, j),\ \psi_i(t_1, x^\ell_{t_1})\Delta t + \hat{E}_{t_1}\left[J^k(t_2, \cdot, i)\right](x^\ell_{t_1})\Big). \qquad (3)$$

The full recursive *pathwise* construction for J^k is

$$J^k(m\Delta t, x^\ell_{m\Delta t}, i) = \begin{cases} \psi_i(m\Delta t, x^\ell_{m\Delta t})\,\Delta t\ + J^k((m+1)\Delta t, x^\ell_{(m+1)\Delta t}, i), & \text{no switch;} \\ -C_{i,j} + J^{k-1}(m\Delta t, x^\ell_{m\Delta t}, j), & \text{switch to } j. \end{cases} \qquad (4)$$

Remarks

- Regression used solely to update the optimal stopping times τ^k
- Regressed values never stored
- Helps to eliminate potential biases from the regression step.

Algorithm

1. Select a set of basis functions (B_j) and algorithm parameters $\Delta t, M^\sharp, N^p, \bar{K}, \delta$.
2. Generate N^p paths of the driving process: $\{x^\ell_{m\Delta t}, m = 0, 1, \ldots, M^\sharp, \ell = 1, 2, \ldots, N^p\}$ with fixed initial condition $x^\ell_0 = x_0$.
3. Initialize the value functions and switching times $J^k(T, x^\ell_T, i) = 0$, $\tau^k(T, x^\ell_T, i) = M^\sharp\ \forall i, k$.
4. Moving backward in time with $t = m\Delta t$, $m = M^\sharp, \ldots, 0$ repeat the Loop:
   - Compute inductively the layers $k = 0, 1, \ldots, \bar{K}$ (evaluate $\mathbb{E}\left[J^k(m\Delta t + \Delta t, \cdot, i) | \mathcal{F}_{m\Delta t}\right]$ by linear regression of $\{J^k(m\Delta t + \Delta t, x^\ell_{m\Delta t + \Delta t}, i)\}$ against $\{B_j(x^\ell_{m\Delta t})\}_{j=1}^{N^B}$, then add the reward $\psi_i(m\Delta t, x^\ell_{m\Delta t}) \cdot \Delta t$)
   - Update the switching times and value functions
5. end Loop.
6. Check whether $\bar{K}$ switches are enough by comparing $J^{\bar{K}}$ and $J^{\bar{K}-1}$ (they should be equal).

Observe that during the main loop we only need to store the buffer $J(t, \cdot), \ldots, J(t + \delta, \cdot)$; and $\tau(t, \cdot), \cdots, \tau(t + \delta, \cdot)$.

Convergence

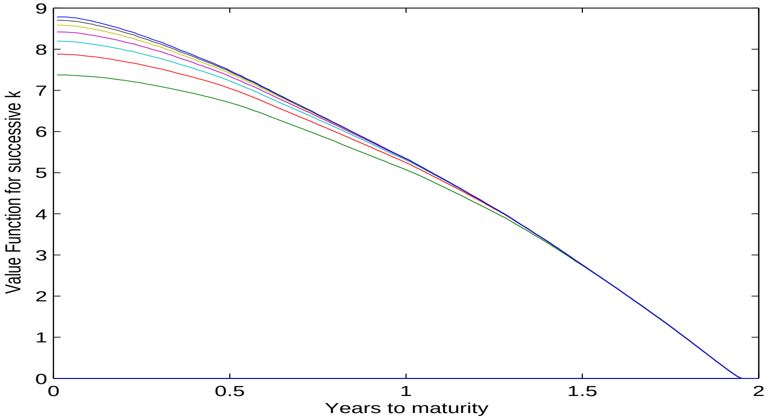
- **Bouchard - Touzi**
- **Gobet - Lemor - Warin**

Example 1

$$dX_t = 2(10 - X_t)\,dt + 2\,dW_t, \qquad X_0 = 10,$$

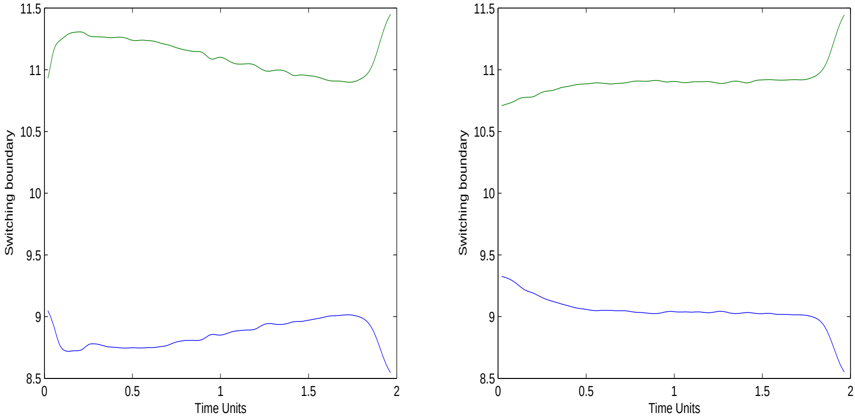
- Horizon $T = 2$,
- Switch separation $\delta = 0.02$.
- Two regimes
- Reward rates $\psi_0(X_t) = 0$ and $\psi_1(X_t) = 10(X_t - 10)$
- Switching cost $C = 0.3$.

Value Functions



$J^k(t, x, 0)$ as a function of t

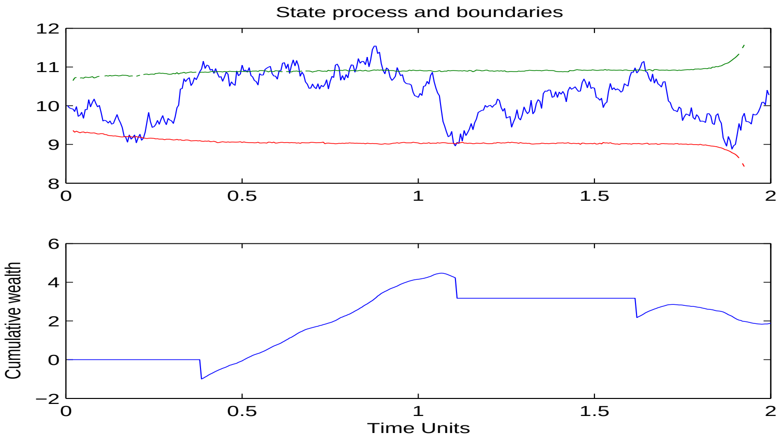
Exercise Boundaries



$k = 2$ (left) $k = 7$ (right)

NB: Decreasing boundary around $t = 0$ is an artifact of the Monte Carlo.

One Sample


State process and boundaries
12
11
10
9
8
0
0.5
1
1.5
2
Cumulative wealth
6
4
2
0
−2
Time Units

Example 2: Comparisons

Spark spread $X_t = (P_t, G_t)$

$$\begin{cases} \log(P_t) \sim OU(\kappa = 2, \theta = \log(10), \sigma = 0.8) \\ \log(G_t) \sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4) \end{cases}$$

- $P_0 = 10$, $G_0 = 10$, $\rho = 0.7$
- Agreement Duration $[0, 0.5]$
- Reward functions

$$\begin{aligned} \psi_0(X_t) &= 0 \\ \psi_1(X_t) &= 10(P_t - G_t) \\ \psi_2(X_t) &= 20(P_t - 1.1\, G_t) \end{aligned}$$

- **Switching costs**

$$C_{i,j} = 0.25|i - j|$$

Numerical Comparison

Method	Mean	Std. Dev	Time (m)
Explicit FD	5.931	–	25
LS Regression	5.903	0.165	1.46
TvR Regression	5.276	0.096	1.45
Kernel	5.916	0.074	3.8
Quantization	5.658	0.013	400^*

Table: Benchmark results for Example 2.

Example 3: Dual Plant & Delay

$$\begin{cases} \log(P_t) \sim OU(\kappa = 2, \theta = \log(10), \sigma = 0.8), \\ \log(G_t) \sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4), \\ \log(O_t) \sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4), . \end{cases}$$

- $P_0 = G_0 = O_0 = 10$, $\rho_{pg} = 0.5$,$\rho_{po} = 0.3$, $\rho_{go} = 0$
- Agreement Duration $T = 1$
- Reward functions

$$\begin{aligned} \psi_0(X_t) &\equiv 0 \\ \psi_1(X_t) &= 5 \cdot (P_t - G_t) \\ \psi_2(X_t) &= 5 \cdot (P_t - O_t), \\ \psi_3(X_t) &= 5 \cdot (3P_t - 4G_t) \\ \psi_4(X_t) &= 5 \cdot (3P_t - 4O_t). \end{aligned}$$

- Switching costs $C_{i,j} \equiv 0.5$
- Delay $\delta = 0, 0.01, 0.03$ (up to ten days)

Numerical Results

Setting	No Delay	$\delta = 0.01$	$\delta = 0.03$
Base Case	13.22	12.03	10.87
Jumps in P_t	23.33	22.00	20.06
Regimes 0-3 only	11.04	10.63	10.42
Regimes 0-2 only	9.21	9.16	9.14
Gas only: $0, 1, 3$	9.53	7.83	7.24

Table: LS scheme with 400 steps and 16000 paths.

Remarks

- High δ lowers profitability by over 20%.
- Removal of regimes: without regimes 3 and 4 expected profit drops from 13.28 to 9.21.

Example 4: Exhaustible Resources

Include I_t current level of resources left (I_t non-increasing process).

$$J(t,x,c,i) = \sup_{\tau,j} \mathbb{E}\Big[\int_t^{\tau} \psi_i(s,X_s)\,ds + J(\tau,X_\tau,I_\tau,j) - C_{i,j} \,|\, X_t = x, I_t = c\Big]. \tag{5}$$

◇ Resource depletion (boundary condition) $J(t,x,0,i) \equiv 0$.
◇ Not really a control problem I_t can be computed ***on the fly***

Mining example of Brennan and Schwartz varying the initial copper price X_0

Method/ X_0	0.3	0.4	0.5	0.6	0.7	0.8
BS '85	1.45	4.35	8.11	12.49	17.38	22.68
PDE FD	1.42	4.21	8.04	12.43	17.21	22.62
RMC	1.33	4.41	8.15	12.44	17.52	22.41

Extension to Gas Storage & Hydro Plants

- Accomodate **outages**
- Include switch separation as a form of **delay**
- Was extended (**R.C. - M. Ludkovski**) to treat
  - **Gas Storage**
  - **Hydro Plants**
- **Porchet-Touzi**

What Remains to be Done

- Need to improve delays
- Need **convergence analysis**
- Need better analysis of **exercise boundaries**
- Need to implement duality upper bounds
  - we have approximate value functions
  - we have approximate exercise boundaries
  - so we have lower bounds
  - need to extend **Meinshausen-Hambly** to optimal switching set-up

Extending the Analysis Adding Access to a Financial Market

Porchet-Touzi

- Same (Markov) factor process $X_t = (X_t^{(1)}, X_t^{(2)}, \cdots)$ as before
- Same plant characteristics as before
- Same operation control $u = (\xi, \mathcal{T})$ as before
- Same maturity T (end of tolling agreement) as before
- **Reward** for operating the plant

$$H(x, i, T; u)(\omega) \triangleq \int_0^T \psi_{u_s}(s, X_s)\, ds - \sum_{\tau_k < T} C(u_{\tau_k -}, u_{\tau_k})$$

Hedging/Investing in Financial Market

Access to a financial market (possibly incomplete)

- y initial wealth
- π_t investment portfolio
- $Y_T^{y,\pi}$ corresponding terminal wealth from investment
- **Utility function** $U(y) = -e^{-\gamma y}$
- Maximum expected utility

$$v(y) = \sup_{\pi} \mathbb{E}\{U(Y_T^{y,\pi})\}$$

- With the power plant (tolling contract)

$$V(x, i, y) = \sup_{u,\pi} \mathbb{E}\{U(Y_T^{y,\pi} + H(x, i, T; u))\}$$

INDIFFERENCE PRICING

$$\overline{p} = p(x, i, y) = \sup\{p \geq 0; V(x, i, y - p) \geq v(y)\}$$

Analysis of

- BSDE formulation
- PDE formulation