Optimal Execution:
IV. Heterogeneous Beliefs and Market Making

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The Agents

Market Maker

- Nasdaq definition: agent that places competitive orders on both sides of the order book in exchange for privileges.
- Acts as a scaled-down version of the market.
- In this lecture: **Liquidity provider**, someone who posts an order book/transaction cost curve.
- Strategy: adapt pricing by *reading client flows*.

Clients

- In this lecture: **Liquidity takers**, agents who trade with the Market maker.
- Are *information* driven.

Theoretical literature

- **Early approaches**: **Hasbrouck(2007), Chakrborti - Toke - Patriarca - Abergel(2011)**
- **Inventory** models: **Garman(1976), Amihud - Mendelson(1980)**
- **Informed trader** models: **Kyle(1985), O'Hara(1995)**
- **Zero-intelligence** models: **Gode - Sunder(1993), Maslov(2000), Cont(2008)**
- **Market impact** models: **Almgren - Chriss(2000), Bouchaud - Potters (2006), Schied(2007)**

Objective

Propose a **stochastic**, ***agent-based*** model in which existence and (*tractable* and *realisitc*) properties of the LOB appear as a result of the analysis (**not as hypotheses**)

Client model

- Summarize sparsely the link between trade and price dynamics.

Market maker model

- *Tractable* market making strategy based on previous result.

R.C. - K. Webster (2012)

Setup: heterogeneous beliefs

Let

1. $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with W a $\mathbb{P}$-BM that generates $\mathbb{F}$.
2. $\mathbb{F}^k \subset \mathbb{F}$ generated by a $\mathbb{P}$-BM W^k.
3. $\mathbb{P}^k$ s.t. $\mathbb{P}^k|_{\mathcal{F}_t^k} \sim \mathbb{P}|_{\mathcal{F}_t^k}$.
4. P_t an Itô process adapted to *all* $\left(\mathbb{F}^k\right)_{k=0\ldots n}$.
5. In L^2 and a.s. P_t grows polynomially in t.

NB

Each agent has his /her own distinct *filtration* and *probability measure*. They are potentially mutually exclusive, but the **price process is adapted to all of them**

Anatomy of a trade

- **Midprice** P_t announced by the market at time t
- **Market maker** proposes a **transaction cost curve** $c_t(I)$ around P_t
- Market maker cannot differentiate clients **pre-trade**
- Client triggers a **trade** of volume I_t
- Client obtains volume I_t and pays **cash flow** $P_t I_t + c_t(I_t)$.
- Market maker tries to identify clients **post-trade**

Setup: transaction costs

Agents behaviors

- Market maker controls transaction **cost function** $l \longmapsto c_t(l)$.
- Client i controls **trading volumes/speeds** l_t^i.

Hypotheses

1. Marginal costs are defined: $c \in C^1$.
2. Clients may choose not to trade, $c_t(0) = 0$ and the midprice is well defined, $c_t'(0) = 0$.
3. Marginal costs increase with volume: c_t is convex.
4. c_t has compact support.

Duality relationship

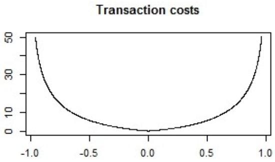
Legendre transform

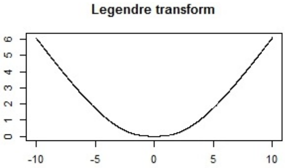
$$\gamma_t(\alpha) := \sup_{l \in \text{supp}(c_t)} (\alpha l - c_t(l))$$

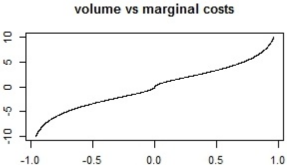
Duality

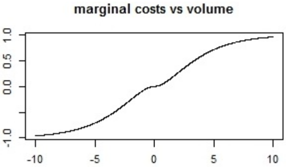
c_t convex with compact support $\iff \gamma_t''$ is a positive finite measure.

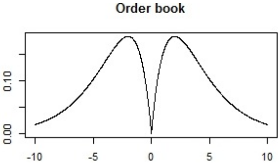
The distribution γ_t'' represents the **order book** formed by the orders of the market maker.


Transaction costs
0
10
30
50
-1.0
-0.5
0.0
0.5
1.0


Legendre transform
0
1
2
3
4
5
6
-10
-5
0
5
10


volume vs marginal costs
-10
-5
0
5
10
-1.0
-0.5
0.0
0.5
1.0


marginal costs vs volume
-1.0
0.0
0.5
1.0
-10
-5
0
5
10


Order book
0.00
0.10
-10
-5
0
5
10

Client model

Client's Objective

- Summarize sparsely the link between trade and price dynamics in a general, theoretical framework.

Not trying to build a optimal trading strategy.

Assumptions

- The client only tries to *predict*, not *cause* price movements.
- The client's decision does not affect c_t.

Realistic if the client is 'small enough'.

Client model

- **Exogeneous state variables** P_t and c_t are Itô processes. P_t has polynomial growth and c_t convex with compact support.
- **Endogeneous state variables**

$$\begin{cases} dL_t^i &= l_t^i dt \\ dX_t^i &= L_t^i dP_t - c_t(l_t^i)dt \end{cases}$$

L_t^i is the *total* position of the client. X_t^i is his *wealth*, marked to the midprice. l_t^i, the rate at which he trades, is his *control*.

- **Objective function**

$$J^i = \mathbb{E}_{\mathbb{P}^i}\left[U^i(X_{\tau^i}^i, p_{\tau^i})\right]$$

with τ^i a stopping time.

Optimal trading strategy

Theorem

Under suitable integrability assumptions on U^i and τ^i, the optimal strategy is

$$\alpha_t^i := c_t'(I_t^i) = \mathbb{E}_{\mathbb{Q}^i}\left[p_{\tau^i} - P_t \middle| \mathcal{F}_t^i \right]$$

with $\frac{d\mathbb{Q}^i}{d\mathbb{P}^i} = \frac{\partial_X U^i(X^i_{\tau^i}, p_{\tau^i})}{\mathbb{E}_{\mathbb{P}^i}\left[\partial_X U^i(X^i_{\tau^i}, p_{\tau^i})\right]}$.

Testing the client model

Hypotheses

- Under $\mathbb{Q}^i$, $\tau^i \sim \exp\left(\beta^i\right)$ independent of P_t.
- $\sigma^i_t := | \underbrace{c'_t(I^i_t)}_{\text{Implied alpha}} - \underbrace{(p_{\tau^i} - P_t)}_{\text{Realized alpha}} | \leq \frac{spread}{2}$

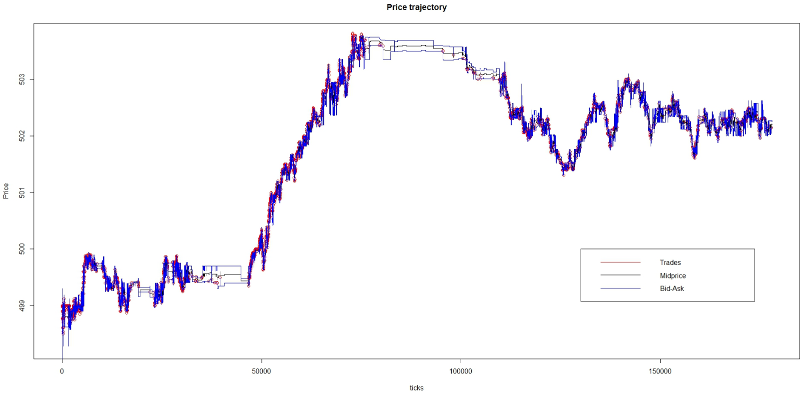
This leads to a *two parameter* model linking trade to price dynamics: (β^i, σ^i).

Testing the hypotheses on data

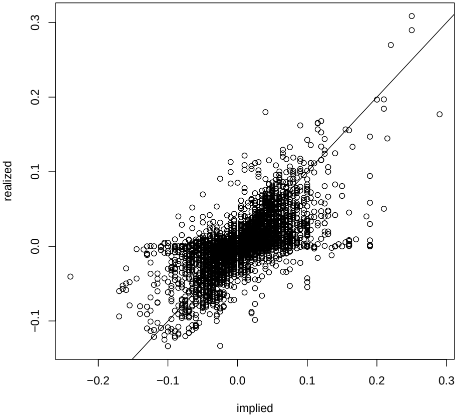
- Assume all clients have one of *two* time scales.
- choose (β_1, β_2) that minimizes error between implied and realized alpha.

Source

- Nasdaq 'fullview' data: all public quotes, all trades, nanosecond timestamps.
- Long parsing time: Data goes from 7:00-10:00am.



Two time scales



- L^1 regression used.
- Time scales: 9 ($\approx$ 0.5 seconds) and 158 ticks.
- Mean error: 0.026.
- Mean half-spread: 0.063.
- Lower bound on error: 0.005.

Market maker model

Market Maker's Objectives

- Find a *tractable* market making strategy based on previous result.
- Build a theoretical model for the order book that *replicates* the empirical features described before.

Strategy

Exploit link between trade and price dynamics to dynamically adapt pricing.

Market maker model: endogenous variables

With primal variables

$$\begin{cases} dL_t & = -\frac{1}{n}\sum_i l_t^i dt \\ dX_t & = L_t dP_t + \frac{1}{n}\sum_i c_t(l_t^i) dt \end{cases}$$

With dual variables

$$\begin{cases} dL_t & = -\frac{1}{n}\sum_i \gamma_t'\left(\alpha_t^i\right) dt \\ dX_t & = L_t dP_t + \frac{1}{n}\sum_i \left[\alpha_t^i \gamma_t'\left(\alpha_t^i\right) - \gamma_t\left(\alpha_t^i\right)\right] dt \end{cases}$$

Assume the market maker is risk-neutral.

Model for the α_t^i

- **Notation**
  We will denote by $\mu_t(\alpha)$ the client belief distribution, that is, the empirically observed distribution of the (α_t^i).
- **Microscopic model(SDE)**

$$d\alpha_t^i = -\rho\alpha_t^i dt + \sigma dB_t^i + \nu dB_t$$

  mean reversion corresponds to *decay of information*.
- **Macroscopic model(SPDE)**

$$d\mu_t(\alpha) = \left[\frac{1}{2}\left(\sigma^2 + \nu^2\right)\Delta\mu_t(\alpha) + \rho\nabla\left(\alpha\mu_t(\alpha)\right)\right] dt - \nu\nabla\mu_t(\alpha)dB_t$$

Approximate model for P_t

- **Intuition**
  - Do not want to make an explicit model for the price process.
  - Instead, would like to *infer* the price from client trades.
- **Implied alpha relationship**

$$\alpha_t^i := c_t'(I_t^i) = \mathbb{E}_{\mathbb{Q}^i}\left[\int_t^\infty e^{-\beta^i(t-s)} dp_s \middle| \mathcal{F}_t^i\right]$$

- **Estimator**

$$dp_t^\lambda := \sum_{i=1}^n \lambda^i \left(\beta^i \alpha_t^i dt - d\alpha_t^i\right)$$

with $\sum \lambda^i = 1$.

Estimation result

Entropic feedback

There exists λ s.t.

$$\mathbb{E}\left|P_t - p_t^\lambda\right|^2 \leq \epsilon^2 \frac{1}{n}\sum_i E(\mathbb{Q}^i, \mathbb{P}) \approx -\epsilon^2 \int_0^t \left\langle \log\left(\frac{\gamma_s''}{\mu_s}\right), \mu_s \right\rangle ds$$

with E the *entropy* function and

$$\epsilon = \sqrt{\frac{n}{\sum_i \left(\sigma^i\right)^{-2}}} \leq \frac{1}{n}\sum_i \sigma^i$$

Approximate control problem

State variables

$$\begin{cases} dL_t & = -\langle \gamma_t', \mu_t \rangle \, dt \\ d\mu_t(\alpha) & = \left[\frac{1}{2}\left(\sigma^2 + \nu^2\right)\Delta\mu_t(\alpha) + \rho\nabla\left(\alpha\mu_t(\alpha)\right)\right] dt - \nu\nabla\mu_t(\alpha) dB_t \end{cases}$$

Objective function

$$J^\lambda = \int_0^\infty e^{-\beta t}\mathbb{E}\left[L_t \langle id, (\beta\lambda)_t \rangle + \langle -L_t\beta id + (id - \bar{\alpha}_t)\gamma_t' - \gamma_t, \mu_t \rangle\right] dt$$

under the constraint $\int_0^\infty \left\langle e^{-\beta t} \log\left(\frac{\gamma_t''}{\mu_t}\right), \mu_t \right\rangle dt \le C.$

Pontryagin

BSDE

The solution to the Pontryagin BSDE gives rise to the market maker's 'shadow alpha':

$$\alpha_t^* = \left\langle id, \lambda_t + \frac{(\beta\lambda)_t - \beta\mu_t}{\beta + \rho} \right\rangle$$

Hamiltonian

$$\mathcal{H}(\gamma, \mu, \alpha^*) = \langle (id - \alpha^*)\gamma' - \gamma + \epsilon \log \gamma'', \mu \rangle$$

Result

Profitability of an order without feedback

Define

$$m(\alpha) = \underbrace{(\alpha - \alpha^*)}_{\textit{spread}} \cdot \underbrace{\int_\alpha^\infty \mu}_{\textit{filling probability}} \quad \textit{if } \alpha \geq 0$$

then we have:

$$\mathcal{H}(\gamma, \mu, \alpha^*) = \langle \gamma'', m \rangle + \epsilon \langle \log \gamma'', \mu \rangle$$

Optimal strategy with feedback

$$\frac{\gamma''(\alpha)}{\mu(\alpha)} = \frac{\epsilon}{C - m(\alpha)}$$

where C is a renormalization constant.

Simulated example

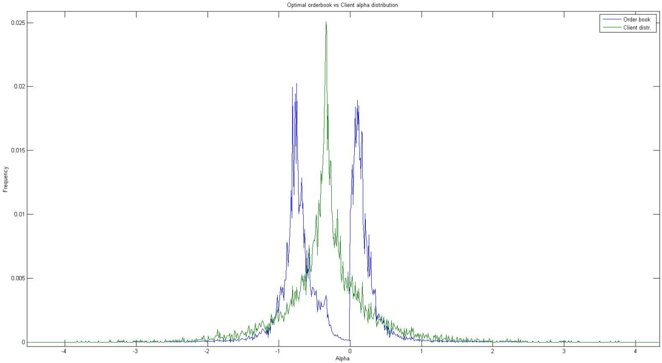


Figure: Blue: Optimal order book γ''. Green: Client alpha distribution μ.